Linear algebra, WNE, 2018/2019
meeting 11. – solutions

8 November 2018

1. Let linear transformation $\varphi, \psi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be such that $\varphi((2,2,1)) = (19,12)$, $\varphi((1,1,0)) = (10,0)$, $\varphi((1,0,0)) = (3,1)$ and $\psi((-1,2,1)) = (5,11), \psi((-1,1,0)) = (4,1), \psi((0,0,1)) = (3,9)$. Find a formula for $\varphi + 3\psi$.

   - $\varphi((1,0,0)) = (3,1)$
   - $(0,1,0) = (1,1,0) - (1,0,0)$, so $\varphi((0,1,0)) = (10,0) - (3,1) = (7,-1)$
   - $(0,0,1) = (2,2,1) - 2(1,1,0)$, thus $\varphi((0,0,1)) = (19,12) - 2(10,0) = (-1,12)$
   - $(1,0,0) = (-1,2,1) - 2(-1,1,0) - (0,0,1)$, therefore $\psi((1,0,0)) = (5,11) - 2(4,1) - (3,9) = (-6,0)$
   - $(0,1,0) = (-1,2,1) - (-1,1,0) - (0,0,1)$, so $\psi((0,1,0)) = (5,11) - (4,1) - (3,9) = (-2,1)$
   - $\psi((0,0,1)) = (3,9)$

   Hence, $\varphi((x,y,z)) = (3x+7y-z, x-y+12z)$ and $\psi((x,y,z)) = (-6x-2y+3z, y+9z)$. So $(\varphi+3\psi)(x,y,z) = (-15x+y+8z, x+2y+39z)$.

2. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$. Find $A+2B$. Find a matrix C, such that $A+C = B$.

$$A + 2B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 6 & -2 \\ 2 & 4 & -2 \\ 4 & 2 & 0 \\ 4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -3 \\ 5 & 6 & -2 \\ 4 & 3 & 0 \\ 5 & 5 & 0 \end{bmatrix}$$

$$C = B - A = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 0 & -1 \\ 2 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

3. Calculate $A \cdot B$ for

   - $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & -1 & -2 \\ 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$
   - $A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \\ 2 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

   - $AB = \begin{bmatrix} 1\cdot 0 + 0 \cdot 1 + 2 \cdot 0 & 1 \cdot 3 + 0 \cdot 2 + 2 \cdot 1 & 1 \cdot (-1) + 0 \cdot 0 + 2 \cdot 0 & 1 \cdot (-2) + 0 \cdot (-1) + 2 \cdot 2 \\ 1 \cdot 0 + 3 \cdot 1 + 1 \cdot 0 & 1 \cdot 3 + 3 \cdot 2 + 1 \cdot 1 & 1 \cdot (-1) + 3 \cdot 0 + 1 \cdot 0 & 1 \cdot (-2) + 3 \cdot (-1) + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & -1 & 2 \\ 3 & 10 & -1 & -3 \end{bmatrix}$

- $AB = \begin{bmatrix} 3\cdot 4+4\cdot 0 & 3\cdot 3+4\cdot 2 & 3\cdot(-1)+4\cdot(-1) \\ 1\cdot 4+3\cdot 0 & 1\cdot 3+3\cdot 2 & 1\cdot(-1)+3\cdot(-1) \\ 2\cdot 4+2\cdot 0 & 2\cdot 3+2\cdot 2 & 2\cdot(-1)+2\cdot(-1) \\ (-1)\cdot 4+0\cdot 0 & (-1)\cdot 3+0\cdot 2 & (-1)\cdot(-1)+0\cdot(-1) \\ 0\cdot 4+1\cdot 0 & 0\cdot 3+1\cdot 2 & 0\cdot(-1)+1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 12 & 17 & -7 \\ 4 & 9 & -4 \\ 8 & 10 & -4 \\ -4 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

4. Check whether for all $A, B \in M_{n\times n}(\mathbb{R})$, $AB = BA$.

   No, e.g. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, ale $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.