Linear algebra, WNE, 2018/2019
meeting 10. – solutions

5 November 2018

1. Which of the following mappings $\varphi\colon V \to W$ are linear?

   - $V = \mathbb{R}^3, W = \mathbb{R}^2, \varphi((x, y, z)) = (x + 3y - 1, 4x + 2y + 6)$,
   - $V = \mathbb{R}^3, W = \mathbb{R}^2, \varphi((x, y, z)) = (x + 3y - z, 4x + 2y + 6z)$,
   - $V = \mathbb{R}^3, W = \mathbb{R}^2, \varphi((x, y, z)) = (x + 3y - z, 4|x| + 2|y| + 6|z|)$,
   - $V = F(\mathbb{R}, \mathbb{R}), W = \mathbb{R}, \varphi(f) = 4f(5) - 5f(4)$.

   - no, because $\varphi(2(0, 0, 0)) = \varphi((0, 0, 0)) = (-1, 6)$, but $2\varphi((0, 0, 0)) = 2(-1, 6) = (-2, 12)$.
   - yes, because for every $g \in \mathbb{R}$ and any $(a, b, c), (d, e, f)$ we get

     $$\varphi((a, b, c) + (d, e, f)) = \varphi((a + d, b + e, c + f)) =$$
     $$(a+d+3b+3e-c-f, 4a+4d+2b+2e+6c+6f) = (a+3b-c, 4a+2b+6c)+(d+3e-f, 4d+2e+6f) =$$
     $$\varphi((a, b, c)) + \varphi((d, e, f))$$

     and $\varphi(g(a, b, c))\varphi((ga, gb, gc)) = (ga + 3gb - gc, 4ga + 2gb + 6gc) = g(a + 3b - c, 4a + 2b + 6c) = g\varphi((a, b, c))$.
   - no, because $\varphi(-1(1, 0, 0)) = \varphi((-1, 0, 0)) = (-1, 4)$, but $(-1)\varphi((1, 0, 0)) = (-1)(1, 4) = (-1, -4)$.
   - yes, because for every $g, h\colon \mathbb{R} \to \mathbb{R}$ and any $a \in \mathbb{R}$ we get $\varphi((g + h)) = 4(g + h)(5) - 5(g + h)(4) = 4g(5) + 4h(5) - 5g(4) - 5h(4) = \varphi(g) + \varphi(h)$ and $\varphi(ag) = 4ag(5) - 5ag(4) = a\varphi(g)$.

2. For which real numbers $t \in \mathbb{R}$ the mapping $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi((a, b)) = (a + b + (t^2 - 9)ab, 5a + 3(b - 1) + t)$ is linear?

   We check the condition on multiplication for 2 and vector $(1, 1)$: $\varphi(2(1, 1)) = \varphi((2, 2)) = (-32+4t^2, 13+t)$ and $2\varphi((1, 1)) = (-14 + 2t^2, 10 + 2t)$. If φ is a linear mapping, those two vectors are equal, and $13 + t = 10 + 2t$, so $t = 3$. It is easy to check, that if $t = 3$ this is a linear mapping.

3. Find the formulas for the following linear mappings.

   - $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3, \varphi((1, 0, 1)) = (5, 1, 3), \varphi((0, 1, 1)) = (2, 3, 4), \varphi((1, 0, 0)) = (6, 7, 7)$,
   - $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3, \varphi((3, 1)) = (4, 5, -1), \varphi((7, 2)) = (-3, 0, 5)$.

   The problem is to find the coordinates of the standard basis in the given basis. In the first example (denote $\alpha = (1, 0, 1), \beta = (0, 1, 1), \gamma = (1, 0, 0)$) we see that $(1, 0, 0) = \gamma$, $(0, 0, 1) = \alpha - \gamma$ and $(0, 1, 0) = -\alpha + \beta + \gamma$. So $\varphi((1, 0, 0)) = (6, 7, 7), \varphi((0, 1, 0)) = -(5, 1, 3) + (2, 3, 4) + (6, 7, 7) = (3, 9, 8)$ and $\varphi((0, 0, 1)) = (5, 1, 3) - (6, 7, 7) = (-1, -6, -4)$, Thus, finally $\varphi((x, y, z)) = (6x + 3y - z, 7x + 9y - 6z, 7x + 8y - 4z)$.

   In the second example we calculate the coordinates of the vectors $(1, 0), (0, 1)$ in the basis $\alpha = (3, 1), \beta = (7, 2)$ in the usual way (two systems of equations in one matrix):

   $$\begin{bmatrix} 3 & 7 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 1 & 0 \end{bmatrix} \xrightarrow{w_2 - 3w_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix} \xrightarrow{w_1 - 2w_2} \begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

   So $(1, 0) = -2\alpha + \beta$ and $(0, 1) = 7\alpha - 3\beta$. Thus, $\varphi((1, 0)) = -2(4, 5, -1) + (-3, 0, 5) = (-11, -10, 7)$ and $\varphi((0, 1)) = 7(4, 5, -1) - 3(-3, 0, 5) = (37, 35, -22)$, so finally $\varphi((x, y)) = (-11x+37y, -10x+35y, 7x-22y)$.

4. Let $\varphi, \psi \colon \mathbb{R}^3 \to \mathbb{R}^2$, be linear mapping such that $\varphi((1,1,1)) = (3,7)$, $\varphi((1,1,0)) = (2,5)$, $\varphi((1,0,0)) = (1,6)$ and $\psi((2,2,1)) = (3,3), \psi((2,1,0)) = (5,0), \psi((2,1,1)) = (4,2)$. Find the formula for $\varphi + \psi$ and 5φ.

   Similarly as before we calculate the expressions for φ i ψ, first finding the coordinates of the vectors from the standard basis. Let $\alpha = (1,1,1), \beta = (1,1,0), \gamma = (1,0,0)$. We see that $(1,0,0) = \gamma, (0,1,0) = \beta - \gamma$ and $(0,0,1) = \alpha - \beta$. So $\varphi((1,0,0)) = (1,6), \varphi((0,1,0)) = (2,5) - (1,6) = (1,-1)$ and $\varphi((0,0,1)) = (3,7) - (2,5) = (1,2)$, thus $\varphi((x,y,z)) = (x+y+z, 6x-y+2z)$.

   We also find the formula for ψ, so we have to find the coordinates of vectors $(1,0,0), (0,1,0), (0,0,1)$ in the basis $\gamma = (2,2,1), \delta = (2,1,0), \epsilon = (2,1,1)$. We use the standard method

$$\begin{bmatrix} 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 2w_1}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 2 & 0 & 1 & 0 & -2 \end{bmatrix} \xrightarrow{w_3 - 2w_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 2 & 1 & -2 & 2 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{1}{2}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & 1 \end{bmatrix} \xrightarrow{w_1 - w_3, w_2 + w_3} \begin{bmatrix} 1 & 0 & 0 & \frac{-1}{2} & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & 1 \end{bmatrix}$$

   So $(1,0,0) = -\frac{\gamma}{2} + \frac{\delta}{2} + \frac{\epsilon}{2}, (0,1,0) = \gamma - \epsilon$ i $(0,0,1) = -\delta + \epsilon$, thus $\psi((1,0,0)) = -\frac{1}{2}(3,3) + \frac{1}{2}(5,0) + \frac{1}{2}(4,2) = \left(3, \frac{1}{2}\right), \psi((0,1,0)) = (3,3) - (4,2) = (-1,1)$ and $\psi((0,0,1)) = -(5,0) + (4,2) = (-1,2)$, so finally $\psi((x,y,z)) = \left(3x - y - z, \frac{x}{2} + y + 2z\right)$.

   Therefore, $(\varphi + \psi)((x,y,z)) = (x+y+z, 6x-y+2z) + \left(3x - y - z, \frac{x}{2} + y + 2z\right) = \left(4x, \frac{13x}{2} + 4z\right)$ and $5\varphi((x,y,z)) = 5(x+y+z, 6x-y+2z) = (5x+5y+5z, 30x-5y+10z)$.