Linear algebra, WNE, 2017/2018
before the first test

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Contents

1 Finding roots of a polynomial

1.1 Method

If the degree of a polynomial > 2 we guess the first root checking the integers near zero. Assume that we have find a root a. Using Bezout's Theorem, we divide (similarly as numbers) the polynomial by $(x - a)$, getting to a polynomial of degree lower by one, and I continue using this polynomial.

When a polynomial of degree two is obtained, we calculate the roots by calculating Δ.

1.2 Example

We find the roots of $w(x) = x^4 - x^3 - 21x^2 + x + 20$. It is easy to see that 1 is a root, indeed $w(1) = 1 - 1 - 21 + 1 + 20 = 0$. Divide $w(x)$ by $(x - 1)$:

$$
\begin{array}{cccccc}
x^3 & +0x^2 & -21x & -20 & & \\
\hline
x^4 & -x^3 & -21x^2 & +x & +20 & :(x-1) \\
x^4 & -x^3 & & & & \\
 & 0x^3 & -21x^2 & & & \\
 & 0x^3 & -0x^2 & & & \\
 & & -21x^2 & +x & & \\
 & & -21x^2 & +21x & & \\
 & & & -20x & +20 & \\
 & & & -20x & +20 & \\
 & & & & 0 &
\end{array}
$$

We get $v(x) = x^3 - 21x - 20$. Again, we guess that -1 is a root. Indeed, $-1 + 21 = 0$. We divide $v(x)$ by $x + 1$.

$$
\begin{array}{ccccc}
x^2 & -x & -20 & & \\
\hline
x^3 & +0x^2 & -21x & -20 & :(x+1) \\
x^3 & +x^2 & & & \\
 & -x^2 & -21x & & \\
 & -x^2 & -x & & \\
 & & -20x & -20 & \\
 & & -20x & -20 & \\
 & & & 0 &
\end{array}
$$

We get $x^2 - x - 20$ and now it is possible to calculate the roots in the usual way. $\Delta = (-1)^2 - 4(-20) = 81$, so $x_1 = \frac{1-9}{2} = -4$ and $x_2 = \frac{1+9}{2} = 5$. So, finally $w(x) = (x-1)(x+1)(x+4)(x-5)$ and the roots are $-1, 1, -4, 5$.

1.3 Exemplary problems

Find the roots of polynomial $x^4 + 8x^3 - x^2 - 68x + 60$.

2 Solving a system of linear equations using Gaussian-elimination method

2.1 Method

We write out a matrix of the considered system by writing the coeffitients and free coefficient in the subsequent rows. Next, we transform the matrix into an echelon form using an appropriate sequence of permitted operations. There are three types of permitted operations:

- adding (or subtracting) to a row another row multiplied by a number,
- swapping two rows,
- multiplying a row by a non-zero number.

An echelon form satisfies the following conditions. All the zero rows should be beneath all the non-zero rows. In every row the first non-zero number (called the leading coefficient) if further to the right than the leading coefficient in the previous row. To achieve an echelon form we obtain zeroes using the operations under subsequent leading coefficients starting from the leftmost column. To obtain zeroes below a leading coefficient use the first-type operation and subtract from a row the row with leading coefficient multiplied by such a number to obtain zero in the considered position. The other two types of operations are used at this stage mainly to simplify the calculations.

After achieving an echelon form we reduce it by multiplying the rows by numbers in such a way to obtain 1 as every leading coefficient. We also have to subtract the rows to obtain zeroes above every leading coefficient. Usually it is best to do this starting with the rightmost column.

Three cases are then possible:

- the system is inconsistent (does not have any solutions). If this is the case, it can be observed in the last non-zero row. This is the case if in this row we have zeroes on the left-hand side of equation and the non-zero number in the column of free coefficients (actually, this is clear already in a non-reduced echelon form).
- the system has exactly one solution, if in every column (except from the column of free coefficients) there is a leading coefficient. It suffices now to write out this solution.
- otherwise the system has infinitely many solutions, and in this case we have to write out the general solution, by leaving all the basic variables (from columns with leading coefficients) on the left-hand side of equations and moving everything else to the right-hand side.

2.2 Examples

1.

$$\begin{cases} 2x+5y-10z=-1 \\ x+3y-2z=0 \\ -x+5z=33 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & 5 & -10 & -1 \\ 1 & 3 & -2 & 0 \\ -1 & 0 & 5 & 33 \end{array}\right] \xrightarrow{w_1 \leftrightarrow w_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 5 & -10 & -1 \\ -1 & 0 & 5 & 33 \end{array}\right] \xrightarrow{w_2-2w_1,\, w_3+w_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -1 & -6 & -1 \\ 0 & 3 & 3 & 33 \end{array}\right] \xrightarrow{w_3+3w_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -1 & -6 & -1 \\ 0 & 0 & -15 & 30 \end{array}\right] \xrightarrow{w_2\cdot(-1),\, w_3\cdot\left(-\frac{1}{15}\right)} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{w_1+2w_3,\, w_2-6w_3}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -4 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{w_1-3w_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -43 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

So this system has exactly one solution, and the solution is $(-43, 13, -2)$.

2.

$$\begin{cases} 2a-3b+5c-d=2 \\ -8a+12b-26c+6d=4 \\ 6a-9b+18c-4d=-4 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 2 & -3 & 5 & -1 & -2 \\ -8 & 12 & -26 & 6 & 4 \\ 6 & -9 & 18 & -4 & -4 \end{array}\right] \xrightarrow{w_2+4w_1,\, w_3-3w_1}$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 5 & -1 & -2 \\ 0 & 0 & -6 & 2 & -4 \\ 0 & 0 & 3 & -1 & 2 \end{array}\right] \xrightarrow{w_2 \leftrightarrow w_3} \left[\begin{array}{ccccc} 2 & -3 & 5 & -1 & -2 \\ 0 & 0 & 3 & -1 & 2 \\ 0 & 0 & -6 & 2 & -4 \end{array}\right] \xrightarrow{w_3+2w_2}$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 5 & -1 & -2 \\ 0 & 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1\cdot\frac{1}{2},\, w_2\cdot\frac{1}{3}} \left[\begin{array}{cccc|c} 1 & -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1-\frac{5}{2}w_2}$$

$$\left[\begin{array}{cccc|c} 1 & -\frac{3}{2} & 0 & \frac{1}{3} & -\frac{8}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} a=-\frac{8}{3}+\frac{3}{2}b-\frac{1}{3}d \\ c=\frac{2}{3}+\frac{1}{3}d \end{cases}$$

So every vector of form $(-\frac{8}{3}+\frac{3}{2}b-\frac{1}{3}d, b, \frac{2}{3}+\frac{1}{3}d, d)$ is a solution for any $b, d \in \mathbb{R}$.

3.

$$\begin{cases} 3x_1+x_2+x_3=6 \\ -2x_1-x_3=-4 \\ 5x_1+3x_2+x_3=10 \\ 6x_1+8x_2-x_3=13 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ -2 & 0 & -1 & -4 \\ 5 & 3 & 1 & 10 \\ 6 & 8 & -1 & 13 \end{array}\right] \xrightarrow{3w_2+2w_1,\, 3w_3-5w_1,\, w_4-2w_1}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 6 & -3 & 1 \end{array}\right] \xrightarrow{w_3+2w_2,\, w_4-3w_2} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_4} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is an echelon form, and there is no need to reduce it further, because we see that this system is inconsistent. There is a contradiction on the last non-zero row $(0 = 1)$.

2.3 Exemplary problems

Solve the following systems of equations.

$$\begin{cases} x_1+2x_2+4x_3+x_4=0 \\ -3x_1+x_2+3x_3+5x_4=0 \\ 5x_1+2x_2+7x_3=0 \end{cases}$$

$$\begin{cases} a-b+c=2 \\ 2b-c=1 \\ -a+b-c=0 \\ -a+8b+7c=-4 \end{cases}$$

$$\begin{cases} 3x+2y+3z=4 \\ x+y+z=8 \\ 5x+3y+6z=9 \end{cases}$$

3 Finding a polynomial given by its values in some points

3.1 Method

If some values of the considered polynomial are given for some arguments, we substitute those arguments under x in the general expression $(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)$, and end up with equations describing coefficients $a_0, \ldots, a_n$. Then, we shall solve the obtained system of equations.

Sometimes, the problem suggests using the Viete'a formulas. For a polynomial of second degree $(ax_2 + bx + c)$ we have:

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 x_2 = \frac{c}{a} \end{cases},$$

which imply equations describing the coefficients if the sum or the product of roots is given. Then, at the end we shall check whether those roots actually exist.

3.2 Example

Does there exist a polynomial $w(x) = ax^2 + bx + c$ of degree two such that $w(-3) = 2$, $w(1) = -2$ and which has both real roots with product -8. If the answer is in the positive find it.

We get the following system of equations.

$$\begin{cases} 9a - 3b + c = 2 \\ a + b + c = -2 \\ 8a + c = 0 \end{cases}.$$

Which has to be solved.

$$\rightarrow \left[\begin{array}{ccc|c} 9 & -3 & 1 & 2 \\ 1 & 1 & 1 & -2 \\ 8 & 0 & 1 & 0 \end{array}\right] \xrightarrow{w_1 \leftrightarrow w_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 9 & -3 & 1 & 2 \\ 8 & 0 & 1 & 0 \end{array}\right] \xrightarrow{w_2 - 9w_1, w_3 - 8w_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -12 & -8 & 20 \\ 0 & -8 & -7 & 16 \end{array}\right] \xrightarrow{w_2 \cdot \frac{1}{4}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & -8 & -7 & 16 \end{array}\right] \xrightarrow{3w_3 - 8w_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & -5 & 8 \end{array}\right] \xrightarrow{w_2 \cdot \left(-\frac{1}{3}\right), w_3 \cdot \left(-\frac{1}{5}\right)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & -\frac{8}{5} \end{array}\right] \xrightarrow{w_1 - w_3, w_2 - \frac{2}{3}w_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{8}{5} \end{array}\right] \xrightarrow{w_1 - w_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{8}{5} \end{array}\right]$$

Therefore, $a = \frac{1}{5}, b = -\frac{3}{5}, c = -\frac{8}{5}$. We check whether the roots actually exist $\Delta = \frac{9}{25} + 4 \cdot \frac{8}{25} = \frac{41}{25} > 0$. This is the case, so $\frac{1}{5}x^2 - \frac{3}{5}x - \frac{8}{5}$ is the considered polynomial.

3.3 Exemplary problems

1. Find a polynomial $w(x)$ of degree 3, such that $w(0) = -1, w(1) = 3, w(2) = 7, w(-1) = 5$.

2. Find a polynomial $w(x)$ of degree 2, such that $w(-1) = 4, w(2) = -2$, with two real roots sum of which equals 2.

4 Finding for which values of a parameter a system of equations is inconsistent or has exactly one solution

4.1 Method

We transform a matrix of the considered system of equations into an echelon form (there is no need to reduce it) and consider what value is is needed. Recall that a system is inconsistent if in an echelon form the last non-zero row has zeroes on the left-hand side of the equality, and a non-zero number as the free coefficient. It has exactly one solution, if it has a leading coefficient in all the columns (except from the column of free coefficients).

4.2 Example

For which real numbers $s, t \in \mathbb{R}$ the system

$$\begin{cases} 3x_1 + x_2 + x_3 = 6 \\ -2x_1 - x_3 = -4 \\ 5x_1 + 3x_2 + x_3 = 10 \\ 6x_1 + (5-t)x_2 - x_3 = s \end{cases}$$

is inconsistent, and for which it has exactly one solution. We transform the matrix into an echelon form:

$$\to \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ -2 & 0 & -1 & -4 \\ 5 & 3 & 1 & 10 \\ 6 & 2-t & -1 & s \end{array}\right] \xrightarrow{3w_2+2w_1, 3w_3-5w_1, w_4-2w_1} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & t & -3 & s-12 \end{array}\right] \xrightarrow{w_3+2w_2, w_4-tw_2}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3-t & s-12 \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_4} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3-t & s-12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system has exactly one solution, if there is a leading coefficient in all the columns (on the left-hand side of equality). Here, this is the case if and only if $t \neq -3$. Then regardless of s the system has exactly one solution. If, on the other hand, $t = -3$, there are two possibilities. Either $s = 12$, and this row is a zero row (an the system has infinitely many solutions), or $s \neq 12$, and then we have a contradiction (and the system is inconsistent). So, finally:

$$\begin{cases} t \neq -3 \Rightarrow \text{there is exactly one solution} \\ t = -3, s \neq 12 \Rightarrow \text{the system is inconsistent,} \\ t = -3, s = 12 \Rightarrow \text{there are infinitely many solutions.} \end{cases}$$

4.3 Exemplary problems

Check for which $s, t \in \mathbb{R}$ the system

$$\begin{cases} 2x + 5y - sz = -1 \\ x + 3y - 2z = 0 \\ -x + 5z = t \end{cases}$$

is inconsistent, and for which it has exactly one solution.

5 Checking whether a vector is a linear combination of a given system of vectors

5.1 Method

We shall check whether a vector $\beta = (b_1, \ldots, b_n)$ is a linear combination of $\alpha_1 = (a_{11}, \ldots a_{1n})$, $\alpha_2 = (a_{21}, \ldots a_{2n}), \ldots, \alpha_k = (a_{k1}, \ldots a_{kn})$. If the anwer is in the positive, there exist coefficients $c_1, \ldots, c_k$ such that $(b_1, \ldots, b_n) = c_1(a_{11}, \ldots a_{1n}) + c_2(a_{21}, \ldots a_{2n}) + \ldots + c_k(a_{k1}, \ldots a_{kn})$. This equality considered on subsequent coordinates, gives the following system of equations:

$$\begin{cases} a_{11}c_1 + a_{21}c_2 + \ldots + a_{k1}c_k = b_1 \\ a_{12}c_1 + a_{22}c_2 + \ldots + a_{k2}c_k = b_2 \\ \ldots \\ a_{1n}c_1 + a_{2n}c_2 + \ldots + a_{kn}c_k = b_n \end{cases}$$

This system consists of n equations, and the coefficients $c_1 \ldots c_k$ are the variables in it, so vector β is a linear combination of the given system of vectors, if the considered system of equations is consistent

In other words, in the subsequent columns we put the vectors $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta$, and we transform the matrix to an echelon form and check whether the last non-zero row generates a contradiction.

5.2 Example

We check whether a vector $(6, -4, 10, 13)$ is a linear combination of $(3, -2, 5, 6), (1, 0, 3, 8), (1, -1, 1, -1)$. Putting those vectors in the columns we get the matrix which was already transformed to an echelon form in the third example in the second topic.

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ -2 & 0 & -1 & -4 \\ 5 & 3 & 1 & 10 \\ 6 & 8 & -1 & 13 \end{array}\right] \rightarrow \ldots \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & 1 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, this is an inconsistent system, and so the considered vector is not a linear combination of the given system of vectors.

5.3 Exemplary problems

1. Check whether the vectors $(1, 2, 1, 2, 1, 3)$, $(7, 7, 1, 4, 1, 1)$ are linear combinations of $(1, 3, 1, 2, 5, 3)$, $(4, 5, 1, 3, 3, 2)$.
2. For which real numbers $r \in \mathbb{R}$ vector $(r, 8, 6)$ is a linear combination of $(3, 4, 5)$, $(1, 4, 4)$, $(7, 4, 7)$?

6 Checking whether a system of vectors is linearly independent

6.1 Methods

6.1.1 First method

A system of vectors is linearly independent, if it is not possible to get a zero vector using a non-trivial linear combination of those vectors. The operation on the rows of matrix are exactly non-trivial linear combinations and when transforming a matrix into an echelon form we will always get a zero row if only it is possible.

Thus, we put the vectors in rows of a matrix, and transform it into an echelon form. The last row is a non-zero row if and only if the system is linearly independent. It is linearly dependent if and only if the last row in an echelon form is a zero row.

6.1.2 Second method

A system of vectors $\alpha_1, \ldots, \alpha_k$ if and only if when $a_1\alpha_1 + \ldots + a_k a_k = 0$, then $a_1 = \ldots = a_k = 0$. In other words if and only if the only coefficients that can be used in a linear combination which gives the zero vector are zeroes. Similarly as before the vector equation $a_1\alpha_1 + \ldots + a_k a_k = 0$ is actually a system of linear equations describing the coefficients $a_1, \ldots, a_k$, if we write out equations for subsequent coordinates. The question is whether the system given by a matrix in which in the subsequent columns we have vectors $\alpha_1, \alpha_2, \ldots, \alpha_k, 0$ has any solution except from the zero solution. In other words, whether this system has more than one solution.

To sum up, to check whether a system of vectors is linearly independent, we put those vectors in columns of a matrix along with zero column of free coefficients, and check whether this system has exactly one solution (whether there is a leading coefficient in every column (except from the column of free coefficients)). If this is the case the system is linearly independent. Otherwise, it is linearly dependent.

6.2 Examples

1. We check whether system $(2,-3,5,-1,-2), (-8,12,-26,6,4), (6,-9,18,-4,-4)$ is linearly independent using the first method (I put the vectors in rows of a matrix). This matrix was already considered in the second topic.

$$\begin{bmatrix} 2 & -3 & 5 & -1 & -2 \\ -8 & 12 & -26 & 6 & 4 \\ 6 & -9 & 18 & -4 & -4 \end{bmatrix} \to \ldots \to \begin{bmatrix} 2 & -3 & 5 & -1 & -2 \\ 0 & 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   We get a row of zeroes, so the system is not linearly independent (it is linearly dependent).

2. We check whether system $(2,1,-1), (5,3,0), (-10,2,5)$ is linearly independent using the second method. We put the vectors in the columns. This matrix was already considered in the second topic.

$$\left[\begin{array}{ccc|c} 2 & 5 & -10 & 0 \\ 1 & 3 & -2 & 0 \\ -1 & 0 & 5 & 0 \end{array}\right] \to \ldots \to \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & 0 & -15 & 0 \end{array}\right]$$

   We get a leading coefficient in all columns (except from the column of free coefficients), thus there is exactly one solution (the zero vector), and so the considered system is linearly independent.

6.3 Exemplary problems

Check whether the following systems of vectors are linearly independent.

- $(1,0,2), (2,3,1), (4,3,5)$,
- $(1,-1,1,1,-2), (4,4,-4,-4,0), (3,1,3,-1,3), (-1,0,1,0,0), (0,-1,0,1,0)$.

7 Finding coordinates of a vector with respect to a given basis

7.1 Metoda

We will find the coordinates of $\beta = (b_1, \ldots, b_n)$ with respect to a basis $\alpha_1 = (a_{11}, \ldots a_{1n}), \alpha_2 = (a_{21}, \ldots a_{2n}), \ldots,$ $\alpha_k = (a_{k1}, \ldots a_{kn})$, so coefficients $c_1, \ldots, c_k$, such that $(b_1, \ldots, b_n) = c_1(a_{11}, \ldots a_{1n}) + c_2(a_{21}, \ldots a_{2n}) + \ldots +$ $c_k(a_{k1}, \ldots a_{kn})$. This equation translates into the system of equations for the coordinates.

$$\begin{cases} a_{11}c_1 + a_{21}c_2 + \ldots + a_{k1}c_k = b_1 \\ a_{12}c_1 + a_{22}c_2 + \ldots + a_{k2}c_k = b_2 \\ \ldots \\ a_{1n}c_1 + a_{2n}c_2 + \ldots + a_{kn}c_k = b_n \end{cases}$$

which is a system of n equations with variables $c_1 \ldots c_k$. Since we consider a basis, we know that this system has exactly one solution. This solution gives the coefficients.

So, to find the coordinates we put in the columns of a matrix the vectors of the considered basis along with the considered vector as the column of free coefficients and we solve this system.

7.2 Example

We find the coordinates of $(-1, 0, 33)$ with respect to basis $(2, 1, -1), (5, 3, 0), (-10, 2, 5)$. The system of equation

$$\begin{cases} 2x + 5y - 10z = -1 \\ x + 3y - 2z = 0 \\ -x + 5z = 33 \end{cases}$$

was already solved in the second topic. The solution is $(-43, 13, -2)$, thus $(-1, 0, 33) = -43(2, 1, -1) +$ $13(5, 3, 0) - 2(-10, 2, 5)$.

7.3 Exemplary problems

Find the coordinates of $(1, 2, 3, -1, 5) \in \mathbb{R}^5$ with respect to the basis $(1, -1, 1, 1, -2)$, $(0, -1, 0, 0, 0)$, $(3, 1, 3, -1, 3)$, $(-1, 0, 1, 0, 0)$, $(0, -1, 0, 1, 0)$.

8 Finding a basis and dimension of a space given as a set of linear combinations

8.1 Method

We notice that all the permitted operations on the rows of a matrix can be reversed. Therefore, doing those operations does not change the set of all linear combinations of rows of a matrix. Additionally, in an echelon form the non-zero rows are linearly independent. This implies the method of finding a basis when given a system of vectors which spans a considered space. We put those vectors in rows of a matrix and transform the matrix into an echelon form. The non-zero rows constitute a basis of the considered space. The number of vectors in a basis is called the dimension.

8.2 Example

We will find a basis and dimension of $\text{lin}((2, -3, 5, -1, -2), (-8, 12, -26, 6, 4), (6, -9, 18, -4, -4))$. The matrix with such rows was already transformed into echelon form in the second topic.

$$\begin{bmatrix} 2 & -3 & 5 & -1 & -2 \\ -8 & 12 & -26 & 6 & 4 \\ 6 & -9 & 18 & -4 & -4 \end{bmatrix} \to \ldots \to \begin{bmatrix} 2 & -3 & 5 & -1 & -2 \\ 0 & 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $(2, -3, 5, -1, -2), (0, 0, 3, -1, 2)$ is a basis, and the dimension equals 2.

8.3 Exemplary problems

Find a basis and dimension of spaces spanned by the following systems of vectors.

- $(1, 2, 1, 3), (2, 5, 4, 4), (1, 3, 3, 1)$,
- $(3, 2, 1, 2), (9, 6, 3, 6), (6, 6, 6, 5), (6, 8, 10, 6)$,
- $(1, 2, 1), (0, 1, 1), (1, 3, 2)$.

9 Find a basis and dimension of a space described by a system of linear equations

9.1 Method

First, we obtain the general solution, and next generate those vectors by substituting 1 under each parameter and zero under the rest of them. Obviously, such a system of vectors generates by linear combinations all possible solutions. Moreover, it is easy to check that such system is linearly independent.

9.2 Example

We will find basis and dimension of a space given by the following system of equations.

$$\begin{cases} 2a - 3b + 5c - d = 0 \\ -8a + 12b - 26c + 6d = 0 \\ 6a - 9b + 18c - 4d = 0 \end{cases}$$

We have solved a similar system in the second topic, and we know that the general solution takes the following form

$$\begin{cases} a = \frac{3}{2}b - \frac{1}{3}d \\ c = \frac{1}{3}d \end{cases}$$

which in parametric form gives $(\frac{3}{2}b - \frac{1}{3}d, b, \frac{1}{3}d, d)$ for $b, d \in \mathbb{R}$. Substituting $b = 1, d = 0$ and $b = 0, d = 1$ we get basic vectors: $(\frac{3}{2}, 1, 0, 0), (-\frac{1}{3}, 0, \frac{1}{3}, 1)$. The dimension equals two.

9.3 Exemplary problems

Find a basis and dimension of spaces described by the following systems of equations:

$$\begin{cases} 5x_1 + 2x_2 + 8x_3 = 0 \\ 6x_1 - 3x_2 - 4x_3 = 0 \\ 7x_1 + 4x_2 + 9x_3 = 0 \\ 4x_1 - 5x_2 + 4x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 + 3x_2 + x_3 + 5x_4 = 0 \\ 2x_1 + 7x_2 + 9x_3 + 2x_4 = 0 \\ 4x_1 + 13x_2 + 11x_3 + 12x_4 = 0 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 0 \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 0 \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 0 \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 0 \end{cases}$$

$$\begin{cases} 6x + 4y + 5z + 2w + 3t = 0 \\ 3x + 2y + 4z + w + 2t = 0 \\ 3x + 2y - 2z + w = 0 \\ 9x + 6y + z + 3w + 2t = 0 \end{cases}$$

10 Finding a system of equations describing a space of linear combinations of a given system of vectors

10.1 Metoda

Consider a system of vectors $\alpha_1 = (a_{11}, \ldots, a_{1n}), \alpha_2 = (a_{21}, \ldots, a_{2n}), \ldots, \alpha_k = (a_{k1}, \ldots, a_{kn})$. We would like to find a system of equations describing the space spanned by $\alpha_1, \ldots, \alpha_k$. The given vectors have to be solutions to every equation in the system which we are looking for. Consider one such equation: $c_1x_1 + \ldots c_nx_n = 0$. We know that $\alpha_1, \ldots, \alpha_k$ are solutions to this equation, so we get a system of equations which has to be satisfied by all the equations in the system we are looking for.

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + \ldots + a_{1n}c_n = 0 \\ a_{21}c_1 + a_{22}c_2 + \ldots + a_{2n}c_n = 0 \\ \ldots \\ a_{k1}c_1 + a_{k2}c_2 + \ldots + a_{kn}c_n = 0 \end{cases}$$

We solve this system, and calculate a basis $\beta_1 = (b_{11}, \ldots b_{1n}), \ldots, \beta_l = (b_{l1}, \ldots, b_{ln})$ of the space of its solutions. Recall that this is the space of possible coefficients of the system of linear equations which we are looking for. Thus, we can write out this system using the vectors from the basis as coefficients in subsequent equations.

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n = 0 \\ b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n = 0 \\ \ldots \\ b_{l1}x_1 + b_{l2}x_2 + \ldots + b_{ln}x_n = 0 \end{cases}$$

This is the system we are looking for, because we know, that any equation which is solved by all the given vectors can be achieved as a linear combination of those equations.

To sum up, we find a basis of a space of solutions of a homogeneous system of equations given by a matrix with the given vectors as its rows. The vectors from the obtained basis are the coefficients in the system we are looking for.

10.2 Example

Find a system of equations describing the space spanned by $(2, -3, 5, -1)$, $(-8, 12, -26, 6)$, $(6, -9, 18, -4)$. We have to solve the following system first

$$\begin{cases} 2a - 3b + 5c - d = 0 \\ -8a + 12b - 26c + 6d = 0 \\ 6a - 9b + 18c - 4d = 0 \end{cases}$$

This system was considered in the ninth topic, and we know that the basis of the space of its solution takes the following form: $(\frac{3}{2}, 1, 0, 0)$, $(-\frac{1}{3}, 0, \frac{1}{3}, 1)$. Therefore, the following system of equations describes the considered space.

$$\begin{cases} \frac{3}{2}x_1 + x_2 = 0 \\ -\frac{1}{3}x_1 + \frac{1}{3}x_3 + x_4 = 0 \end{cases}$$

10.3 Exemplary problems

Find systems of equations describing the following linear subspaces

$$\text{lin}((2, 1, 4), (3, 5, -1), (3, -2, 13), (7, 7, 7), (-4, -9, 6)),$$

$$\text{lin}((3, 2, 1, 1), (5, 0, 2, 3), (4, 1, 4, 5), (4, 1, -1, -1)),$$

$$\text{lin}((2, 7, -1, 2, 6), (3, 1, 4, 2, 2), (4, -5, 9, 2, -2), (5, 15, 2, 6, 14)).$$

11 Completing if possible a basis of a given subspace uso=ing only vectors from another given subspace

11.1 Method

Assume that we are given two subspaces $V, W \subseteq \mathbb{R}^n$ along with their bases. We would like to complete the basis of V to a basis of $\mathbb{R}^n$ using vectors from W. So, we put all the vectors into rows of a matrix, first the vectors from the basis of V, and below we put the vectors from the basis of W. Next, we try to transform this matrix into echelon form, but without using the operation of swapping the rows. It may be that the final form will be an echelon form, but with some of the rows swapped. If we get less than n non-zero rows, then it is not possible to complete the basis in this way (bases of V and W together do not span the whole space). If , on the other hand, there are n non-zero rows, we take to the basis which we are constructing the rows of the original matrix corresponding to those rows.

11.2 Example

Let $(1,-1,1,1,-2),(4,4,-4,-4,0),(3,1,3,-1,3)$ be a basis of V, and $(-1,0,1,0,0),(0,-1,0,1,0),$ $(-1,0,0,0,1)$ be a basis of W. We complete the basis of V to a basis of $\mathbb{R}^5$ using only vectors from W. We put those vectors in a matrix and transform it:

$$\begin{bmatrix} 1 & -1 & 1 & 1 & -2 \\ 4 & 4 & -4 & -4 & 0 \\ 3 & 1 & 3 & -1 & 3 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 4 & 0 & -4 & 9 \\ 0 & -1 & 2 & 1 & -2 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 0 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 1 & 1 & -2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -4 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We get 5 non-zero rows, so it is possible to complete the basis. The vectors which complete the basis V to a basis of $\mathbb{R}^5$ are
$(-1,0,1,0,0),(0,-1,0,1,0)$.

11.3 Exemplary problems

Let $(-1,0,1,0,0),(0,-1,0,1,0),(-1,0,0,0,1)$ be a basis of W. Check whether it is possible to complete the system $(4,3,2,-3,-6),(1,1,-4,-1,3),(2,0,3,0,-5)$ to a basis of $\mathbb{R}^5$ using only vectors from W.