Linear algebra, WNE, 2017/2018
Exemplary problems before the first test

29 października 2018

All the systems of equations should be solved using Gaussian-elimination method (echelon form).

Problem 1

Find a polynomial $w(x)$ of degree 3 such that $w(1) = -28, w(-2) = 32, w(3) = 72$ with all three real roots such that their product equals 12. Find those roots.

Hint: use the Vieta's formula, which says that the products of the roots of polynomial $ax^3 + bx^2 + cx + d$ equals $-\frac{d}{a}$.

Solution

We get the following system of equations:

$$\begin{cases} a+b+c+d = -28 \\ -8a+4b-2c+d = 32 \\ 27a+9b+3c+d = 72 \\ 12a+d = 0 \end{cases}$$

We use the reverse column order (d, c, b, a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 1 & -2 & 4 & -8 & 32 \\ 1 & 3 & 9 & 27 & 72 \\ 1 & 0 & 0 & 12 & 0 \end{array}\right] \xrightarrow{w_2 - w_1, w_3 - w_1, w_4 - w_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & -3 & 3 & -9 & 60 \\ 0 & 2 & 8 & 26 & 100 \\ 0 & -1 & -1 & 11 & 28 \end{array}\right] \xrightarrow{w_2 \cdot \frac{-1}{3}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & 1 & -1 & 3 & -20 \\ 0 & 2 & 8 & 26 & 100 \\ 0 & -1 & -1 & 11 & 28 \end{array}\right] \xrightarrow{w_3 - 2w_2, w_4 + w_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & 1 & -1 & 3 & -20 \\ 0 & 0 & 10 & 20 & 140 \\ 0 & 0 & -2 & 14 & 8 \end{array}\right] \xrightarrow{w_3 \cdot \frac{1}{10}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & 1 & -1 & 3 & -20 \\ 0 & 0 & 1 & 2 & 14 \\ 0 & 0 & -2 & 14 & 8 \end{array}\right] \xrightarrow{w_4 + 2w_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & 1 & -1 & 3 & -20 \\ 0 & 0 & 1 & 2 & 14 \\ 0 & 0 & 0 & 18 & 36 \end{array}\right] \xrightarrow{w_4 \cdot \frac{1}{18}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -28 \\ 0 & 1 & -1 & 3 & -20 \\ 0 & 0 & 1 & 2 & 14 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{w_1 - w_4, w_2 - 3w_4, w_3 - 2w_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -30 \\ 0 & 1 & -1 & 0 & -26 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{w_1 - w_3, w_2 + w_3}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -40 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{w_1 - w_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -24 \\ 0 & 1 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

So we get the following polynomial: $w(x) = 2x^3 + 10x^2 - 16x - 24$.

We guess the first root 2. We divide the polynomial by $(x - 2)$ i and get $2x^2 + 14x + 12$. Now we use the usual method to obtain the other roots. $\Delta = 100$, so the roots are -6 and -1, so finally, all the roots of $w(x)$ are $-6, -1$ and 2.

Problem 2.

For which real numbers $s, t \in \mathbb{R}$ the following system of equations:

$$\begin{cases} 3x_1 + x_2 + 5x_3 = 10 \\ -x_2 - 2x_3 = -4 \\ (2-t)x_1 + x_2 + 6x_3 = s \\ x_1 + x_2 + 3x_3 = 6 \end{cases}$$

has exactly one solution? For which $s, t \in \mathbb{R}$ it is inconsistent. For $s, t \in \mathbb{R}$ such that the system has exactly one solution, find (depending on s, t) a vector which is its solution. Give an example of a basis of $\mathbb{R}^3$, such that this vector has coordinates $1, -1, -1$ with respect to it (assume that $s \neq 12$).

Solution

So we write down the matrix and transform it into echelon form. We put the columns in order x_2, x_3, x_1 .

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ -1 & -2 & 0 & -4 \\ 1 & 6 & 2-t & s \\ 1 & 3 & 1 & 6 \end{array}\right] \xrightarrow{w_2 + w_1, w_3 - w_1, w_4 - w_1} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 0 & 3 & 3 & 6 \\ 0 & 1 & -1-t & s-10 \\ 0 & -2 & -2 & -4 \end{array}\right] \xrightarrow{w_2 \cdot \frac{1}{3}}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1-t & s-10 \\ 0 & -2 & -2 & -4 \end{array}\right] \xrightarrow{w_3 - w_2, w_4 + 2w_2} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2-t & s-12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If $t \neq -2$, there is a leading coefficient in the last column, so there is only one solution then. On the other hand, if $t = -2$ and $s \neq 12$, then the system is inconsistent. Finally, if $t = -2$ and $s = 12$, the system is consistent but has infinitely many solutions.

We continue assuming $t \neq -2$.

$$\left[\begin{array}{cccc} 1 & 5 & 3 & 10 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2-t & s-12 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1 - 5w_2, w_3 \cdot \frac{1}{-2-t}} \left[\begin{array}{cccc} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{s-12}{-2-t} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1 + 2w_3, w_2 - w_3}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & \frac{2s-24}{-2-t} \\ 0 & 1 & 0 & 2 + \frac{-s+12}{-2-t} \\ 0 & 0 & 1 & \frac{s-12}{-2-t} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The vector which satisfies this system (the solution) is

$$\left(\frac{s-12}{-2-t}, \frac{2s-24}{-2-t}, 2 + \frac{-s+12}{-2-t}\right),$$

and $(\frac{s-12}{-2-t}, 0, 0), (0, \frac{-2s+24}{-2-t}, 0), (0, 0, -2 - \frac{-s+12}{-2-t})$, is a basis (if $s \neq 12$) which satisfies the condition formulated in the problem.

Problem 3.

Let W a subspace of $\mathbb{R}^4$ defined by the following system of equations

$$\begin{cases} x - 3y + z - t = 0 \\ 3x - 9y + 4z + 2t = 0 \end{cases},$$

and let V be a subspace spanned by vectors $(6, 0, -5, 1)$ and $(0, 4, 10, -2)$.

- Find a basis of W and check for which $r \in \mathbb{R}$ it is possible to complete the system of vectors $(1, 0, 0, -2)$, $(-2, 0, 0, r)$ to the basis of the whole $\mathbb{R}^4$ using only vectors from W.
- Give an example of such a basis for $r = 0$ and find coordinates of $(1, 0, -1, 3)$ with respect to this basis.
- Check, whether $W = V$.

Solution

We find the general solution of the system describing W:

$$\begin{bmatrix} 1 & -3 & 1 & -1 \\ 3 & -9 & 4 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

So $(3y+6t, y, -5t, t)$ is the general solution, and thus $\dim W = 2$, and $(3,1,0,0),(6,0,-5,1)$ is a basis of W.

A basis of $\mathbb{R}^4$ consists of 4, so the basis of W and the two given vectors have to be linearly independent, if one can find such a basis of $\mathbb{R}^4$. The two given vectors are independent, if $r \neq 4$. Secondly, each of them need to fail at least one of the equations describing W:

- vector $(1,0,0,-2)$ does not satisfy $x - 3y + z - t = 0$, OK
- vector $(-2,0,0,r)$ does not satisfy $x - 3y + z - t = 0$ for $r \neq -2$ and does not satisfy $3x - 9y + 4z + 2t = 0$ for $r \neq 3$, so for any r it does not satisfy one of the equations, OK.

Thus, for $r \neq 4$ those two vectors along with the basis of W give a basis of the whole $\mathbb{R}^4$.

For $r = 0$ this basis takes the following form: $(1,0,0,-2),(-2,0,0,0),(3,1,0,0),(6,0,-5,1)$. We calculate the coordinates of $(1,0,-1,3)$ with respect to this basis:

$$\begin{bmatrix} 1 & -2 & 3 & 6 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -1 \\ -2 & 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{w_4 + 2w_1} \begin{bmatrix} 1 & -2 & 3 & 6 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -1 \\ 0 & -4 & 6 & 13 & 5 \end{bmatrix} \xrightarrow{w_2 \leftrightarrow w_4, w_3 \leftrightarrow w_4}$$

$$\begin{bmatrix} 1 & -2 & 3 & 6 & 1 \\ 0 & -4 & 6 & 13 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -1 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{-1}{4}, w_4 \cdot \frac{-1}{5}}$$

$$\begin{bmatrix} 1 & -2 & 3 & 6 & 1 \\ 0 & 1 & \frac{-3}{2} & \frac{-13}{4} & \frac{-5}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{w_1 - 6w_4, w_2 + \frac{13}{4}w_4} \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{-1}{5} \\ 0 & 1 & \frac{-3}{2} & 0 & \frac{-3}{5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{w_1 - 3w_3, w_2 + \frac{3}{2}w_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & \frac{-1}{5} \\ 0 & 1 & 0 & 0 & \frac{-3}{5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix} \xrightarrow{w_1 + 2w_2} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-7}{5} \\ 0 & 1 & 0 & 0 & \frac{-3}{5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} \end{bmatrix}$$

So the coordinates are: $\frac{-7}{5}, \frac{-3}{5}, 0, \frac{1}{5}$, and indeed,

$$\frac{-7}{5}(1,0,0,-2) + \frac{-3}{5}(-2,0,0,0) + 0(3,1,0,0) + \frac{1}{5}(6,0,-5,1) = (1,0,-1,3).$$

For $V = W$ we need $\dim V = \dim W$, and this is always the case. Therefore, $V = W$ if and only if $(6,0,-5,1)$ and $(0,4,10,-2)$ are elements of W, i.e. if these vectors satisfy the equations describing W:

- vector $(6,0,-5,1)$ satisfies both the equations,
- vector $(0,4,10,-2)$ satisfies both the equations, as well.

Thus, $V = W$.

Problem 4.

Find a basis and dimension of a subspace of $\mathbb{R}^5$ spanned by $(4,-1,-5,2,7)$, $(1,1,-1,-1,1)$, $(2,1,3,-6,0)$, $(1,0,0,-1,1)$. Next, find a system of equations which describes this space.

Solution

We transform into echelon form the following matrix

$$\begin{bmatrix} 4 & -1 & -5 & 2 & 7 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 1 & 3 & -6 & 0 \\ 1 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_4} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & 1 & 3 & -6 & 0 \\ 4 & -1 & -5 & 2 & 7 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 - 2w_1, w_4 - 4w_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & -4 & -2 \\ 0 & -1 & -5 & 6 & 3 \end{bmatrix} \xrightarrow{w_3 - w_2, w_4 + w_2} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & -4 & -2 \\ 0 & 0 & -6 & 6 & 3 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{1}{4}}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{-1}{2} \\ 0 & 0 & -6 & 6 & 3 \end{bmatrix} \xrightarrow{w_4 + 6w_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So this space is three-dimensional, and $(1, 0, 0, -1, 1), (0, 1, -1, 0, 0), (0, 0, 1, -1, \frac{-1}{2})$ is a basis.

We continue the transformation to get the reduced echelon form (now the matrix should be seen as a system of equation describing possible coefficients of the system of equations which we are looking for):

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 + w_3} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & \frac{-1}{2} \\ 0 & 0 & 1 & -1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution describing the coefficients is: $(a_4 - a_5, a_4 + \frac{a_5}{2}, a_4 + \frac{a_5}{2}, a_4, a_5)$, so $(1, 1, 1, 1, 0), (-2, 1, 1, 0, 2)$ is a basis of the space of possible coefficients. We get the following system of equations.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ -2x_1 + x_2 + x_3 + 2x_5 = 0 \end{cases}$$