Linear algebra, WNE, 2018/2019
meeting 8. – solutions

25 October 2018

1. Let V and W be subspaces of $\mathbb{R}^5$ such that $V = \text{lin}((10, 3, 9+s, 1, 2-s), (4, 1, 6, 1, 1), (2, 1, -1, -1, -2))$, and W is the subspace of solutions of

$$\begin{cases} 3x_1 - 11x_2 + tx_3 - 8x_4 + x_5 = 0 \\ 2x_1 - 4x_2 - x_3 + 3x_4 - x_5 = 0 \\ x_1 - 5x_2 + x_3 - 6x_4 + x_5 = 0 \end{cases}$$

Find $\dim V$ and $\dim W$ depending on $s, t \in \mathbb{R}$. Find s, t such that $V = W$.

We find a basis and dimension V:

$$\begin{bmatrix} 10 & 3 & 9+s & 1 & 2-s \\ 4 & 1 & 6 & 1 & 1 \\ 2 & 1 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} 2 & 1 & -1 & -1 & -2 \\ 4 & 1 & 6 & 1 & 1 \\ 10 & 3 & 9+s & 1 & 2-s \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 5w_1}$$

$$\begin{bmatrix} 2 & 1 & -1 & -1 & -2 \\ 0 & -1 & 8 & 3 & 5 \\ 0 & -2 & 14+s & 6 & 12-s \end{bmatrix} \xrightarrow{w_3 - 2w_2} \begin{bmatrix} 2 & 1 & -1 & -1 & -2 \\ 0 & -1 & 8 & 3 & 5 \\ 0 & 0 & -2+s & 0 & 2-s \end{bmatrix}$$

So if $s = 2$, then $\dim V = 2$, and otherwise $\dim V = 3$.

No let find a basis and dimension of W. We solve the system.

$$\begin{bmatrix} 3 & -11 & t & -8 & -1 \\ 2 & -4 & -1 & 3 & 1 \\ 1 & -5 & 1 & -6 & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} 1 & -5 & 1 & -6 & -1 \\ 2 & -4 & -1 & 3 & 1 \\ 3 & -11 & t & -8 & 1 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 3w_1}$$

$$\begin{bmatrix} 1 & -5 & 1 & -6 & 1 \\ 0 & 6 & -3 & 15 & -3 \\ 0 & 4 & t-3 & 10 & -2 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{3}} \begin{bmatrix} 1 & -5 & 1 & -6 & 1 \\ 0 & 2 & -1 & 5 & -1 \\ 0 & 4 & t-3 & 10 & -2 \end{bmatrix} \xrightarrow{w_3 - 2w_2, w_1 + w_2}$$

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 2 & -1 & 5 & -1 \\ 0 & 0 & t-1 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot (-1)} \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 1 & -5 & 1 \\ 0 & 0 & t-1 & 0 & 0 \end{bmatrix}$$

There are two cases: $t = 1$ and $t \neq 1$. If $t = 1$, then the general solution (with basis variables x_1 and x_3) takes form $(3x_2 + x_4, x_2, 2x_2 + 5x_4 - x_5, x_4, x_5)$, so $\dim W = 3$, and $(3, 1, 2, 0, 0), (1, 0, 5, 1, 0), (0, -1, 0, 1)$ is a basis.

On the other hand, if $t \neq 1$, then:

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 1 & -5 & 1 \\ 0 & 0 & t-1 & 0 & 0 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{1}{t-1}} \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 1 & -5 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 - w_3} \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & -2 & 0 & -5 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

And the general solution (with basic variables x_1, x_3 and x_5) takes form $(3x_2 + x_4, x_2, 0, x_4, 2x_2 + 5x_4)$, so then $\dim W = 2$ and $(3, 1, 0, 0, 2), (1, 0, 0, 1, 5)$ is a basis.

When $V = W$? Firstly, they need to have equal dimensions $\dim V = \dim W$, so we shall consider two cases:

- $\dim V = \dim W = 2$, then $s = 2, t \neq 1$. Next, we check whether the vectors from V satisfy the equations describing W. This is not the case, because they do not satisfy $x_3 = 0$, so in this case $V \neq W$.

- $\dim V = \dim W = 3$, then $s \neq 2, t = 1$. We check for which s the vectors from V satisfy the equations describing W:
  - the first vector: $(2, 1, -1, -1, -2)$, $2 \cdot 1 + 1 \cdot (-3) + (-1) \cdot 0 + (-1) \cdot (-1) + (-2) \cdot 0 = 2 - 3 + 0 + 1 + 0 = 0$ and $2 \cdot 0 + 1 \cdot (-2) + (-1) \cdot 1 + (-1) \cdot (-5) + (-2) \cdot 1 = 0 - 2 - 1 + 5 - 2 = 0$ is ok.
  - the second vector: $(0, -1, 8, 3, 5)$ $0 \cdot 1 + (-1) \cdot (-3) + 8 \cdot 0 + 3 \cdot (-1) + 5 \cdot 0 = 3 - 3 = 0$ and $0 \cdot 0 + (-1) \cdot (-2) + 8 \cdot 1 + 3 \cdot (-5) + 5 \cdot 1 = 2 + 8 - 15 + 5 = 0$, is ok.
  - the third vector: $(0, 0, -2 + s, 0, 2 - s)$ $0 \cdot 1 + 0 \cdot (-3) + (-2 + s) \cdot 0 + 0 \cdot (-1) + (2 - s) \cdot 0 = 0$ and $0 \cdot 0 + 0 \cdot (-2) + (-2 + s) \cdot 1 + 0 \cdot (-5) + (2 - s) \cdot 1 = -2 + s + 2 - s = 0$, is ok as well (for every s).

Hence, $V = W$ if and only if $s \neq 2$ and $t = 1$.

2. Let V be the space described by the following system of equations:

$$\begin{cases} x + y + z + t + w = 0 \\ x - y + z - t + w = 0 \end{cases}$$

Complete, if it is possible, the following systems of vectors to the basis of $\mathbb{R}^5$ using only vectors from V.

- $(5, -1, 2, 1, 7), (2, 3, -6, -3, 4)$,
- $(1, 2, 3, -2, -4), (6, 4, -5, -4, -1), (3, -2, -14, 2, 11)$.

In the first case we have no choice. We need 3 vectors, so we have to take the whole basis of V (V is a tree-dimensional space). This operation is successful if and only if none of the given vectors belongs to V. This is not the case, vector $(2, 3, -6 - 3, 4) \in V$, do it is not possible to complete the system in the required way.
In the second case the situation is even worse, because all 3 vectors belong to V, so it is impossible as well.

3. Find systems of equations describing the following linear subspaces:

- $\text{lin}((4, 1, 2, -3), (2, 3, 1, -9), (2, -1, 1, 3), (6, 4, 3, -12))$,
- $\text{lin}((5, 1, 9, 0, 2), (5, 2, -2, 5, -1), (4, 1, 5, 1, 1))$.

The first subspace: we solve the system describing possible coefficients

$$\begin{bmatrix} 4 & 1 & 2 & -3 \\ 2 & 3 & 1 & -9 \\ 2 & -1 & 1 & 3 \\ 6 & 4 & 3 & -12 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 2 & 3 & 1 & -9 \\ 4 & 1 & 2 & -3 \\ 2 & -1 & 1 & 3 \\ 6 & 4 & 3 & -12 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - w_1, w_4 - 3w_1}$$

$$\begin{bmatrix} 2 & 3 & 1 & -9 \\ 0 & -5 & 0 & 15 \\ 0 & -4 & 0 & 12 \\ 0 & -5 & 0 & 15 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{-1}{15}} \begin{bmatrix} 2 & 3 & 1 & -9 \\ 0 & 1 & 0 & -3 \\ 0 & -4 & 0 & 12 \\ 0 & -5 & 0 & 15 \end{bmatrix} \xrightarrow{w_1 - 3w_2, w_3 + 4w_2, w_4 + 5w_2} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the general solution is $(a_1, 3a_4, -2a_1, a_4)$ and dimension is 2, so we will have two equations. And $(1, 0, -2, 0), (0, 3, 0, 1)$ is a basis, so this is the following system:

$$x_1 - 2x_3 = 03x_2 + x_4 = 0$$

The second space.

$$\begin{bmatrix} 2 & 0 & 9 & 1 & 5 \\ -1 & 5 & -2 & 2 & 5 \\ 1 & 1 & 5 & 1 & 4 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} -1 & 5 & -2 & 2 & 5 \\ 2 & 0 & 9 & 1 & 5 \\ 1 & 1 & 5 & 1 & 4 \end{bmatrix} \xrightarrow{w_2 + 2w_1, w_3 - w_1}$$

$$\begin{bmatrix} -1 & 5 & -2 & 2 & 5 \\ 0 & 10 & 5 & 5 & 15 \\ 0 & 6 & 3 & 3 & 9 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{10}, w_1 \cdot (-1)} \begin{bmatrix} 1 & -5 & 2 & -2 & -5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 6 & 3 & 3 & 9 \end{bmatrix} \xrightarrow{w_3 - 6w_2, w_1 + 5w_2}$$

$$\begin{bmatrix} 1 & 0 & \frac{9}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $(a_1, a_2, a_3, -\frac{3}{2}a_1 - \frac{1}{2}a_2 - \frac{1}{2}a_3, -\frac{5}{2}a_1 - \frac{1}{2}a_2 - \frac{9}{2}a_3)$, so the dimension of the space of coefficients 3 and $(2, 0, 0, -3, -5), (0, 2, 0, -1, -1), (0, 0, 2, -1, -9)$ is a basis, so the following is the system describes the space:

$$\begin{cases} 2x_1 - 3x_4 - 5x_5 = 0 \\ 2x_2 - x_4 - x_5 = 0 \\ 2x_3 - x_4 - 9x_5 = 0 \end{cases}$$

4. Find an example of a vector $\alpha \in \{(x, y, z) \in \mathbb{R}^3 \colon x + 2y - z = 0\}$, such that $(2, 1, 3), (1, 4, 5), \alpha$ is a basis of $\mathbb{R}^3$.

   $(-2, 1, 0), (1, 0, 1)$ is a basis of the space described by the equation and it is easy to check that the first of them is independent from the two given, so we can take it as α.

5. Does there exist $\beta \in \text{lin}((3, 5, 8), (-1, 3, 2))$ such that $(2, 1, 3), (1, 4, 5), \beta$ is a basis of $\mathbb{R}^3$? If so find an example of such vector β.

   Such a vector does not exist, because $(3, 8, 5) = (2, 1, 3) + (1, 4, 5)$ i $(-1, 3, 2) = (1, 4, 5) - (2, 1, 3)$, so $\text{lin}((3, 5, 8), (-1, 3, 2))$ and $\text{lin}((2, 1, 3), (1, 4, 5))$ describe the same space.