Linear algebra, WNE, 2018/2019
meeting 7. – solutions

23 October 2018

1. Find a basis and dimension of a space spanned by vectors $(3,2,1,1),(5,0,2,3),(4,1,4,5),(4,1,-1,-1)$.

   We reverse the column order for easy calculations:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 2 & 0 & 5 \\ 5 & 4 & 1 & 4 \\ -1 & -1 & 1 & 4 \end{bmatrix} \xrightarrow{w_2-3w_1, w_3-5w_1, w_4+w_1} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -6 & -4 \\ 0 & -1 & -9 & -11 \\ 0 & 0 & 3 & 7 \end{bmatrix} \xrightarrow{w_3-w_2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -6 & -4 \\ 0 & 0 & -3 & -7 \\ 0 & 0 & 3 & 7 \end{bmatrix} \xrightarrow{w_4+w_3} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -6 & -4 \\ 0 & 0 & -3 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So the system of vectors
   $(3,2,1,1),(-4,-6,-1,0),(-7,-3,0,0)$ is a basis, and the dimension is 3.

2. Find a system of equation which describes the above space.

   We finish solving the system of equations (we already have an echelon form):

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & -6 & -4 \\ 0 & 0 & -3 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1+w_2, w_3\cdot\frac{-1}{3}} \begin{bmatrix} 1 & 0 & -4 & -1 \\ 0 & -1 & -6 & -4 \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1+4w_3, w_2+6w_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{25}{3} \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2\cdot(-1)} \begin{bmatrix} 1 & 0 & 0 & \frac{25}{3} \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So the general solution is (remember about the reverse order of columns) $(a, -\frac{7}{3}d, 10a, -\frac{25}{3}a)$, and so there is only one vector in a basis of the space of coefficients $(3,-7,30,-25)$, and thus we have ony one equation in the system which we are looking for:

$$3x_1 - 7x_2 + 30x_3 - 25x_4 = 0$$

3. Find a basis and dimension of the space of solutions to the following system of equations.

$$\begin{cases} 5a+10b+6c+3d=0 \\ 2a+4b+4c+3d=0 \\ 3a+6b+5c+5d=0 \end{cases}$$

   We solve the following system of equations (I do not write the columns of zero on the right-hand side):

$$\begin{bmatrix} 5 & 10 & 6 & 3 \\ 2 & 4 & 4 & 3 \\ 3 & 6 & 5 & 5 \end{bmatrix} \xrightarrow{w_1\leftrightarrow w_2} \begin{bmatrix} 2 & 4 & 4 & 3 \\ 5 & 10 & 6 & 3 \\ 3 & 6 & 5 & 5 \end{bmatrix} \xrightarrow{w_2\cdot 2, w_3\cdot 2} \begin{bmatrix} 2 & 4 & 4 & 3 \\ 10 & 20 & 12 & 6 \\ 6 & 12 & 10 & 10 \end{bmatrix} \xrightarrow{w_2-5w_1, w_3-3w_1}$$

$$\begin{bmatrix} 2 & 4 & 4 & 3 \\ 0 & 0 & -8 & -9 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{w_2 \leftrightarrow w_3} \begin{bmatrix} 2 & 4 & 4 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -8 & -9 \end{bmatrix} \xrightarrow{w_3 - 4w_2, w_1 + 2w_2}$$

$$\begin{bmatrix} 2 & 4 & 0 & -5 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 7 \end{bmatrix} \xrightarrow{w_1 \cdot \frac{1}{2}, w_2 \cdot \frac{-1}{2}, w_3 \cdot \frac{1}{7}} \begin{bmatrix} 1 & 2 & 0 & \frac{-5}{2} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{w_1 + \frac{5}{2}w_3, w_2 - 2w_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So we get the general solution: $(-2b, b, 0, 0)$, the dimension is 1, and $(-2, 1, 0, 0)$ is a basis.

4. Find the coordinates of $(5, 0, 0)$ with respect to the basis $(1, 2, -1), (1, 0, 2), (0, 1, 1)$.

   We solve the following system of equations:

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 2 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 + w_1} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -2 & 1 & -10 \\ 0 & 3 & 1 & 5 \end{bmatrix} \xrightarrow{w_3 \cdot 2} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -2 & 1 & -10 \\ 0 & 6 & 2 & 10 \end{bmatrix} \xrightarrow{w_3 + 3w_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -2 & 1 & -10 \\ 0 & 0 & 5 & -20 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{-1}{2} . w_3 \cdot \frac{1}{5}} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix} \xrightarrow{w_2 + \frac{1}{2}w_3} \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix} \xrightarrow{w_1 - w_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

   So the coordinates are $2, 3, -4$, and indeed: $2(1, 2, -1) + 3(1, 0, 2) - 4(0, 1, 1) = (5, 0, 0)$.

5. Let W be a space described by the following system of equations:

$$\begin{cases} x_1 + x_2 + 2x_3 - x_4 + x_5 = 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 - x_5 = 0 \end{cases}$$

   Find a basis of W. Complete the basis to a basis of $\mathbb{R}^5$.

   We solve this system of equations:

$$\begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 2 & 3 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{w_2 - 2w_1} \begin{bmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & 1 & -5 & 4 & -3 \end{bmatrix} \xrightarrow{w_1 - w_2} \begin{bmatrix} 1 & 0 & 7 & -5 & 4 \\ 0 & 1 & -5 & 4 & -3 \end{bmatrix}$$

   So the general solution is $(-7x_3+5x_4-4x_5, 5x_3-4x_4+3x_5, x_3, x_4, x_5)$, thus this space is three-dimensional and $(-7, 5, 1, 0, 0), (5, -4, 0, 1, 0), (-4, 3, 0, 0, 1)$ is a basis. It can be easily seen that along with $(1, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0)$ we can quickly get to an echelon form without the zero row, so this is a basis of $\mathbb{R}^5$.

6. Find an example of a basis of $\mathbb{R}^3$ such that the coordinates of vector $(1, 2, 3)$ with respect to this basis are $3, 1, 2$.

   It suffices to notice that it is enough to change the order of coordinates, so we just use the vectors from the standard basis, just in a different order, i.e.: $(0, 0, 1), (1, 0, 0), (0, 1, 0)$.