Linear algebra, WNE, 2018/2019
meeting 7. – homework solution

23 October 2018

Group 13:45

1. Find a basis and dimension of a space described by the following system of equations.

$$\begin{cases} 2x + 12y + 9z = 0 \\ 8x + 12y + 10z = 0 \\ x - 3y - 2z = 0 \end{cases}$$

We put the columns in order x, y, z and transform the matrix into an echelon form:

$$\begin{bmatrix} 2 & 12 & 9 \\ 8 & 12 & 10 \\ 1 & -3 & -2 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} 1 & -3 & -2 \\ 8 & 12 & 10 \\ 2 & 12 & 9 \end{bmatrix} \xrightarrow{w_2 - 8w_1, w_3 - 2w_1}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 36 & 26 \\ 0 & 18 & 13 \end{bmatrix} \xrightarrow{w_3 - \frac{1}{2}w_4}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 36 & 26 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{36}} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & \frac{13}{18} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + 3w_2} \begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{13}{18} \\ 0 & 0 & 0 \end{bmatrix}$$

So the general solution is the following: $(-\frac{1}{6}z, -\frac{13}{18}z, z)$. Thus the space is one-dimensional, and $\{(-3, -13, 18)\}$ is a basis.

2. Complete system of vectors $(0, 1, 1, 2, 0), (1, 0, 0, 1, 0), (1, 0, 1, 1, 0)$ to a basis of $\mathbb{R}^5$ using only vectors from the space W from Problem 5.

It is easy to see (by reducing a matrix to an echelon form), that by taking the two last vectors from the basis of W, i.e. $(-4, 3, 0, 0, 1)$ (definitely we shall take this vector because all the others have zero as the last coordinate) and $(5, -4, 0, 1, 0)$ we end up with a linearly independent system. Thus, this system is a basis of $\mathbb{R}^5$.

3. Find a system of linear equations describing the space $\mathrm{lin}((-9, 3, 6, -3), (-4, 2, 1, 5), (-5, 5, 4, 3))$.

The system of equations describing the possible coefficients (order of columns: a_4, a_3, a_2, a_1) is:

$$\begin{bmatrix} -3 & 6 & 3 & -9 \\ 5 & 1 & 2 & -4 \\ 3 & 4 & 5 & -5 \end{bmatrix} \xrightarrow{w_1 \cdot \frac{-1}{3}} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 5 & 1 & 2 & -4 \\ 3 & 4 & 5 & -5 \end{bmatrix} \xrightarrow{w_2 - 5w_1, w_3 - 3w_1}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 11 & 7 & -19 \\ 0 & 10 & 8 & -14 \end{bmatrix} \xrightarrow{w_3 \cdot 11} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 11 & 7 & -19 \\ 0 & 110 & 88 & -154 \end{bmatrix} \xrightarrow{w_3 - 10w_2}$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 11 & 7 & -19 \\ 0 & 0 & 18 & 36 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{1}{18}} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 11 & 7 & -19 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{w_1 + w_3, w_2 - 7w_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 11 & 0 & -33 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{11}} \begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{w_1 + 2w_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So the general solution is $(a_1, -2a_1, 3a_1, a_1)$, thus we get a one-dimensional space of coefficients, with $\{(1, -2, 3, 1)\}$ being a basis. Therefore, we get only one equation in the system we are looking for:

$$x_1 - 2x_2 + 3x_3 + x_4 = 0$$

Group 15:00

1. Find a basis and dimension of a space described by the following system of equations.

$$\begin{cases} 9x + 12y + 2z = 0 \\ 5x + 6y + 4z = 0 \\ 2x + 3y - z = 0 \end{cases}$$

We put columns in order z, y, x and transform the matrix to an echelon form:

$$\begin{bmatrix} 2 & 12 & 9 \\ 4 & 6 & 5 \\ -1 & 3 & 2 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_3} \begin{bmatrix} -1 & 3 & 2 \\ 4 & 6 & 5 \\ 2 & 12 & 9 \end{bmatrix} \xrightarrow{w_2 + 4w_1, w_3 - 2w_1} \begin{bmatrix} -1 & 3 & 2 \\ 0 & 18 & 13 \\ 0 & 18 & 13 \end{bmatrix} \xrightarrow{w_3 - w_4, w_1 \cdot (-1)}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 18 & 13 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{18}} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & \frac{13}{18} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + 3w_2} \begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{13}{18} \\ 0 & 0 & 0 \end{bmatrix}$$

So we get the following general solution: $(x, -\frac{13}{18}x, -\frac{1}{6}x)$, and the space is one-dimensional with basis $\{(18, -13, -3)\}$.

2. Complete the system of vectors $(1, 0, 0, 1, 0), (0, 1, 0, 2, 1), (1, 0, 0, 1, 1)$ to a basis of $\mathbb{R}^5$ using only vectors from W from Problem 5.

It is easy to see (by reducing a matrix to an echelon form), that by taking two first vectors from the basis of W, i.e. $(-7, 5, 1, 0, 0)$ (definitely we shall take this vector because all the others have zero as the third coordinate) and $(5, -4, 0, 1, 0)$ we end up with a linearly independent system. Thus, this system is a basis of $\mathbb{R}^5$.

3. Find a system of linear equations which describes the space $\text{lin}((3, 1, 2, -1), (4, 2, 1, 5), (5, 5, 4, 3))$.

We get the following system describing the possible coordinates (column order: a_4, a_3, a_2, a_1):

$$\begin{bmatrix} -1 & 2 & 1 & 3 \\ 5 & 1 & 2 & 4 \\ 3 & 4 & 5 & 5 \end{bmatrix} \xrightarrow{w_2 + 5w_1, w_3 + 3w_1} \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 11 & 7 & 19 \\ 0 & 10 & 8 & 14 \end{bmatrix} \xrightarrow{w_3 \cdot 11} \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 11 & 7 & 19 \\ 0 & 110 & 88 & 154 \end{bmatrix} \xrightarrow{w_3 - 10w_2}$$

$$\begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 11 & 7 & 19 \\ 0 & 0 & 18 & -36 \end{bmatrix} \xrightarrow{w_1 \cdot (-1), w_3 \cdot \frac{1}{18}} \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 11 & 7 & 19 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{w_1 + w_3, w_2 - 7w_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & -5 \\ 0 & 11 & 0 & 33 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{11}} \begin{bmatrix} 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{w_1 + 2w_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

So we get the following general solution $(a_1, 2a_1, -3a_1, -a_1)$, and so the space of coefficients is one-dimensional with basis $\{(1, 2, -3, -1)\}$. Hence the system we are looking for has only one equation:

$$x_1 + 2x_2 - 3x_3 - x_4 = 0$$