Linear algebra, WNE, 2018/2019
meeting 5. – solutions

16 October 2018

1. Does there exist a vector $\alpha \in \mathbb{R}^4$ which is a linear combination of vectors $(1, 1, -1, -2), (1, 0, -3, 1)$ and at the same time a linear combination of vectors $(1, 2, 1, 1), (0, 1, 2, 1)$.

   Let us denote those vectors as v_1, v_2, w_1, w_2. We want to know whether there exist a, b, c, d with at least one of them non-zero such that $av_1 + bv_2 = cw_1 + dw_2$, so $av_1 + bv_2 - cw_1 - dw_2 = 0$, which is a system of linear equation. We have to check whether it has a non-zero solution.

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & -1 & 0 \\ -1 & -3 & -1 & -2 & 0 \\ -2 & 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 + w_1, w_4 + 2w_1} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & -2 & -2 & -2 & 0 \\ 0 & 3 & -3 & -1 & 0 \end{bmatrix} \xrightarrow{w_3 - 2w_2, w_4 + 3w_2}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -4 & 0 \end{bmatrix} \xrightarrow{w_3 \leftrightarrow w_4, w_1 + w_2} \begin{bmatrix} 1 & 0 & -2 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -6 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot (-1), w_3 \cdot \frac{-1}{6}}$$

$$\begin{bmatrix} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + 2w_3, w_2 - w_3} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Thus, the general solution is $(-\frac{1}{3}d, -\frac{1}{3}d, -\frac{2}{3}d, d)$, so there are non-zero solutions, e.g. $(-1, -1, -2, 3)$, hence $-v_1 - v_2 = (-2, -1, 4, 1) = -2w_1 + 3w_2$ is an example of such a vector.

2. Check whether $(5, 1, 1, 3)$ is a linear combination of vectors $(1, 2, 4, 1), (7, 5, 9, 5)$.

   We have to examine the following system of equations.

$$\begin{bmatrix} 1 & 7 & 5 \\ 2 & 5 & 1 \\ 4 & 9 & 1 \\ 1 & 5 & 3 \end{bmatrix} \xrightarrow{w_2 - 2w_1, w_3 - 4w_1, w_4 - w_1} \begin{bmatrix} 1 & 7 & 5 \\ 0 & -9 & -9 \\ 0 & -19 & -19 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow{u_2 \cdot \frac{-1}{9}, u_3 \cdot \frac{-1}{19}, u_4 \cdot \frac{-1}{2}}$$

$$\begin{bmatrix} 1 & 7 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{w_1 - 7w_2, w_3 - w_2, w_4 - w_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   This this system is consistent, and $-2(1, 2, 4, 1) + (7, 5, 9, 5) = (5, 1, 1, 3)$.

3. Check whether the system of vectors $(1, 4, 1), (0, 1, 1), (1, 0, -1)$ is linearly independent.

   We use the second method.

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{w_3 - w_1} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & -4 & -2 \end{bmatrix} \xrightarrow{w_3 + 4w_2} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

   There are no zero rows, so the system is linearly independent.

4. For which real numbers $r \in \mathbb{R}$ vector $(r, 8, 6)$ is a linear combination of vectors $(3, 4, 5), (1, 4, 4), (7, 4, 7)$?

   We write those vectors in order $(1, 4, 4), (3, 4, 5), (7, 4, 7)$ for convenience, and check the system of equations

$$\begin{bmatrix} 1 & 3 & 7 & r \\ 4 & 4 & 4 & 8 \\ 4 & 5 & 7 & 6 \end{bmatrix} \xrightarrow{w_2 - 4w_1, w_3 - 4w_2} \begin{bmatrix} 1 & 3 & 7 & r \\ 0 & -8 & -24 & 8-4r \\ 0 & -7 & -21 & 6-4r \end{bmatrix} \xrightarrow{w_2 \cdot \frac{-1}{8}}$$

$$\begin{bmatrix} 1 & 3 & 7 & r \\ 0 & 1 & 3 & -1+\frac{r}{2} \\ 0 & -7 & -21 & 6-4r \end{bmatrix} \xrightarrow{w_3 + 7w_2} \begin{bmatrix} 1 & 3 & 7 & r \\ 0 & 1 & 3 & -1+\frac{r}{2} \\ 0 & 0 & 0 & -1-\frac{r}{2} \end{bmatrix}$$

   Thus this system is consistent (and the vector is a linear combination) only if $-1 - \frac{r}{2} = 0$, hence if $r = -2$.

5. For which real numbers $s, t \in \mathbb{R}$ vectors $(5, 7, s, 2), (1, 3, 2, 1), (2, 2, 4, t)$ are linearly independent?

   We use the second method

$$\begin{bmatrix} 5 & 7 & s & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 2 & 4 & t \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 5 & 7 & s & 2 \\ 2 & 2 & 4 & t \end{bmatrix} \xrightarrow{w_2 - 5w_1, w_3 - 2w_1} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -8 & s-10 & -3 \\ 0 & -4 & 0 & t-2 \end{bmatrix} \xrightarrow{w_2 \leftrightarrow w_3}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & t-2 \\ 0 & -8 & s-10 & -3 \end{bmatrix} \xrightarrow{w_3 - 2w_2} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -4 & 0 & t-2 \\ 0 & 0 & s-10 & 1-2t \end{bmatrix}$$

   The last row is a non-zero row, if $s \neq 10$ or $t \neq \frac{1}{2}$. The system is linearly independent if at least one of those conditions is met.

6. Let $W = \text{lin}((2, 1, 4), (3, 5, -1), (3, -2, 13), (7, 7, 7), (-4, -9, 6))$. Find a linearly independent system of vectors $\alpha_1, \ldots \alpha_n$ such that $W = \text{lin}(\alpha_1, \ldots, \alpha_n)$

   We transform the following matrix into an echelon form.

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & -1 \\ 3 & -2 & 13 \\ 7 & 7 & 7 \\ -4 & -9 & 6 \end{bmatrix} \xrightarrow{w_2 \cdot 2, w_3 \cdot 2, w_4 \cdot 2} \begin{bmatrix} 2 & 1 & 4 \\ 6 & 10 & -2 \\ 6 & -4 & 16 \\ 14 & 14 & 14 \\ -4 & -9 & 6 \end{bmatrix} \xrightarrow{w_2 - 3w_1, w_3 - 3w_1, w_4 - 7w_1, w_5 + 4w_1}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 7 & -14 \\ 0 & -7 & 14 \\ 0 & 7 & -14 \\ 0 & -7 & 14 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{1}{7}} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -2 \\ 0 & -7 & 14 \\ 0 & 7 & -14 \\ 0 & -7 & 14 \end{bmatrix} \xrightarrow{w_3 + 7w_2, w_4 - 7w_2, w_5 + 7w_2} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   So, for example, the system $(2, 1, 4), (0, 1, -2)$ has the desired properties.