Linear algebra, WNE, 2018/2019
meeting 4. – solutions

11 October 2018

1. Check whether the following subsets of $\mathbb{R}^2$ satisfy any of the conditions from the definition of vector subspace.

   - $\{(x,y)\colon x,y\in\mathbb{Z}\}$,
   - $\{(x,y)\colon |x|-|y|=1\}$.

   In the first case the addition condition is fulfilled ($(x,y)+(z+w)=(x+z,y+w)$ and $x+z,y+w\in\mathbb{Z}$), but does not satisfy the multiplication condition, e.g. $(1,0)$ is an element of this subset, but $\frac{1}{2}(1,0)=(\frac{1}{2},0)$ is not.

   In the second case the addition condition does not hold, because $(1,0),(-1,0)$ are in this set, but $(1,0)+(-1,0)=(0,0)$ is not. The multiplication condition is not satisfied, because $(1,0)$ is an element of this subset, but $\frac{1}{2}(1,0)=(\frac{1}{2},0)$ is not.

2. For which real numbers $s\in\mathbb{R}$ set $W=\{(x,y,z,w)\in\mathbb{R}^4\colon x-2y+z+w=s^2-1 \text{ and } x+y+sw^2=w^2\}$ is a vector subspace?

   Assume that $s<1$. Then vector $v=(1,0,s^2-2-\frac{1}{\sqrt{1-s}},\frac{1}{\sqrt{1-s}})$ satisfies both the equations, but $2v$ does not satisfy the second equation, thus this is not a subspace.

   Assume that $s>1$. Then vector $w=(-1,0,s^2-\frac{1}{\sqrt{s-1}},\frac{1}{\sqrt{s-1}})$ satisfies both the equations, but $2w$ does not satisfy the second equation, thus this is not a subspace.

   Thus, it suffices to check $s=1$. Then actually the set is a linear subspace.

3. Is $(1,1,1,1)\in\mathbb{R}^4$ a linear combination of $(1,2,4,3),(0,1,3,3),(1,2,1,5)$?

   We have to check whether the following system is consistent:

$$\begin{cases} a+c=1\\ 2a+b+2c=1\\ 4a+3b+c=1\\ 3a+3b+5c=1\end{cases}.$$

   We transform its matrix into echelon form.

$$\left[\begin{array}{ccc|c}1&0&1&1\\2&1&2&1\\4&3&1&1\\3&3&5&1\end{array}\right]\xrightarrow{w_2-2w_1,w_3-4w_1,w_4-3w_1}\left[\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&3&-3&-3\\0&3&2&-2\end{array}\right]\xrightarrow{w_3-3w_2,w_4-3w_2}$$

$$\left[\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&0&-3&0\\0&0&2&1\end{array}\right]\xrightarrow{w_3\cdot\frac{-1}{3}}\left[\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&0&1&0\\0&0&2&1\end{array}\right]\xrightarrow{w_4-2w_3}\left[\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{array}\right]$$

   which is a contradiction. So this vector is not a combination of the given vectors.

4. Let $\alpha_1=(3,2,1,1),\alpha_2=(2,7,2,1),\alpha_3=(1,3,1,3)$ and $\beta_1=(2,-2,0,3),\beta_2=(1,1,1,1)$, $\beta_3=(-1,3,1,10)$. Which of vectors β_i are linear combinations of system of vectors $\alpha_1,\alpha_2,\alpha_3$?

This question is equivalent to consistency of three systems of equations with the same left-hand side part, and different free coefficients. In th following matrix the first of those system is represented by columns 1-4th, the second by columns 1-3 and 5th, and the third by columns 1-3 and 6th.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 2 & 1 & -1 \\ 2 & 7 & 3 & -2 & 1 & 3 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 3 & 1 & 10 \end{array}\right] \xrightarrow{w_1 \leftrightarrow w_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 2 & 7 & 3 & -2 & 1 & 3 \\ 3 & 2 & 1 & 2 & 1 & -1 \\ 1 & 1 & 3 & 3 & 1 & 10 \end{array}\right] \xrightarrow{w_2 - 2w_1, w_3 - 3w_1, w_4 - w_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & -2 & -1 & 1 \\ 0 & -4 & -2 & 2 & -2 & -4 \\ 0 & -1 & 2 & 3 & 0 & 9 \end{array}\right] \xrightarrow{w_2 \leftrightarrow w_4} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 & 0 & 9 \\ 0 & -4 & -2 & 2 & -2 & -4 \\ 0 & 3 & 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{w_3 - 4w_2, w_4 + 3w_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 & 0 & 9 \\ 0 & 0 & -10 & -10 & -2 & -40 \\ 0 & 0 & 7 & 7 & -1 & 28 \end{array}\right] \xrightarrow{w_3 \cdot \frac{-1}{10}, w_4 \cdot \frac{1}{7}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 & \frac{1}{5} & 4 \\ 0 & 0 & 1 & 1 & -\frac{1}{7} & 4 \end{array}\right] \xrightarrow{w_4 - w_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 3 & 0 & 9 \\ 0 & 0 & 1 & 1 & \frac{1}{5} & 4 \\ 0 & 0 & 0 & 0 & -\frac{12}{35} & 0 \end{array}\right]$$

So the first and the third systems are consistent, but the second is inconsistent, so β_1 are β_3 linear combinations, but β_2 is not.

5. Is $(1, 2, -1, 2), (1, 4, 2, 8), (-1, 0, 4, 4)$ a linearly independent system of vectors?

   We check whether in the echelon form we get a row of zeroes.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & 4 & 2 & 8 \\ -1 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 + w_1} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 2 & 3 & 6 \\ 0 & 2 & 3 & 6 \end{bmatrix} \xrightarrow{w_3 - w_2} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   We get a row of zeroes, so this system is not linearly independent.