Linear algebra, WNE, 2018/2019
meeting 3. – solutions

9 October 2018

1. Find the general solutions to the following systems of equations.

$$\begin{cases} 3a+2b+3c+4d=8 \\ a+b+c+2d=4 \\ 5a+3b+6c+3d=9 \end{cases}$$

$$\begin{cases} 3a+2b+c+4d+3e=1 \\ 5a+8b+2c+5d+8e=4 \\ 4a-2b+c+7d+e=2 \end{cases}$$

$$\begin{cases} 5x+2y+8z=1 \\ 6x-3y-4z=4 \\ 7x+4y+9z=6 \\ 4x-5y-4z=-2 \end{cases}$$

(a)

$$\begin{bmatrix} 3 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 4 \\ 5 & 3 & 6 & 3 & 9 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 1 & 1 & 1 & 2 & 4 \\ 3 & 2 & 3 & 4 & 8 \\ 5 & 3 & 6 & 3 & 9 \end{bmatrix} \xrightarrow{w_2-3w_1, w_3-5w_1}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 4 \\ 0 & -1 & 0 & -2 & -4 \\ 0 & -2 & 1 & -7 & -11 \end{bmatrix} \xrightarrow{w_3-2w_2} \begin{bmatrix} 1 & 1 & 1 & 2 & 4 \\ 0 & -1 & 0 & -2 & -4 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix} \xrightarrow{w_1+w_2}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & -4 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix} \xrightarrow{w_2\cdot(-1), w_1-w_3} \begin{bmatrix} 1 & 0 & 0 & 3 & 3 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -3 & -3 \end{bmatrix}.$$

So the general solution is:

$$\begin{cases} a=3-3d \\ b=4-2d \\ c=-3+3d \end{cases},$$

in parametric form we get: $(3-3d, 4-2d, -3+3d, d)$.

(b) We write out columns in order: c, a, b, d, e:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 3 & 1 \\ 2 & 5 & 8 & 5 & 8 & 4 \\ 1 & 4 & -2 & 7 & 1 & 2 \end{bmatrix} \xrightarrow{w_2-2w_1, w_3-w_1} \begin{bmatrix} 1 & 3 & 2 & 4 & 3 & 1 \\ 0 & -1 & 4 & -3 & 2 & 2 \\ 0 & 1 & -4 & 3 & -2 & 1 \end{bmatrix} \xrightarrow{w_3+w_2}$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 3 & 1 \\ 0 & -1 & 4 & -3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

so this system is inconsistent.

(c) We write out columns in order: y, x, z:

$$\begin{bmatrix} 2 & 5 & 8 & 1 \\ -3 & 6 & -4 & 4 \\ 4 & 7 & 9 & 6 \\ -5 & 4 & -4 & -2 \end{bmatrix} \xrightarrow{w_2 \cdot 2, w_4 \cdot 2} \begin{bmatrix} 2 & 5 & 8 & 1 \\ -6 & 12 & -8 & 8 \\ 4 & 7 & 9 & 6 \\ -10 & 8 & -8 & -4 \end{bmatrix} \xrightarrow{w_2 + 3w_1, w_3 - 2w_1, w_4 + 5w_1}$$

$$\begin{bmatrix} 2 & 5 & 8 & 1 \\ 0 & 27 & 16 & 11 \\ 0 & -3 & -7 & 4 \\ 0 & 33 & 32 & 1 \end{bmatrix} \xrightarrow{w_2 \leftrightarrow w_3} \begin{bmatrix} 2 & 5 & 8 & 1 \\ 0 & -3 & -7 & 4 \\ 0 & 27 & 16 & 11 \\ 0 & 33 & 32 & 1 \end{bmatrix} \xrightarrow{w_3 + 9w_2, w_4 + 11w_2}$$

$$\begin{bmatrix} 2 & 5 & 8 & 1 \\ 0 & -3 & -7 & 4 \\ 0 & 0 & -47 & 47 \\ 0 & 0 & -45 & 45 \end{bmatrix} \xrightarrow{w_3 \cdot \frac{-1}{47}, w_4 \cdot \frac{-1}{45}} \begin{bmatrix} 2 & 5 & 8 & 1 \\ 0 & -3 & -7 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{w_4 - w_3}$$

$$\begin{bmatrix} 2 & 5 & 8 & 1 \\ 0 & -3 & -7 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 - 8w_3, w_2 + 7w_3} \begin{bmatrix} 2 & 5 & 0 & 9 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 \cdot \frac{-1}{3}}$$

$$\begin{bmatrix} 2 & 5 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 - 5w_2} \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 \cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So this system has exactly one solution, and it is $y = 2, x = 1, z = -1$, in other words $(1, 2, -1)$.

2. Zenobi bought a doughnut, 2 bottles of water and a sandwich, and paid 4,5PLN, and Zdzisław, who bought 2 doughnuts, 5 bottles of water and 3 sandwiches, spent 11PLN. Is it possible to calculate how much money Zbigniew spent, if he bought 2 doughnuts, 6 bottles of water and 4 sandwiches?

   If the purchases of Zbigniew is a sum of a times purchases of Zenobi and b times the purchases of Zdzisław for some real numbers a, b. Translating it into three equations related to doughnuts, water and sandwiches respectively, we get a system of equations.

$$\begin{cases} a + 2b = 2 \\ 2a + 5b = 6 \\ a + 3b = 4 \end{cases}$$

   Such a, b exist if this system of equations is consistent. It is easy to see that it is the case. The solution is $a = -2, b = 2$, so Zbigniew paid $-2 \cdot 4,5 + 2 \cdot 11 = -9 + 22 = 13$ PLN.