Linear algebra, WNE, 2018/2019
meeting 2. – solutions

4 October 2018

1. Which of the following systems of equations are homogeneous? Which are inconsistent? Which have exactly one solution?

$$U_1: \begin{cases} 2x_1 - x_2 = 1 \\ x_1 + 2x_2 = 8 \end{cases},$$

$$U_2: \begin{cases} x_1 + 2x_2 + 4x_3 + x_4 = 0 \\ -3x_1 + x_2 + 3x_3 + 5x_4 = 0 \\ 5x_1 + 2x_2 + 7x_3 = 0 \end{cases},$$

$$U_3: \begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_2 - x_3 = 8 \\ -x_1 + x_2 - x_3 = 0 \\ -x_1 + 8x_2 + 7x_3 = -4 \end{cases},$$

$$U_4: x_1 + 2x_2 - x_3 + x_4 = 5.$$

U_1 is neither homogeneous nor inconsistent, and it has exactly one solution (the solution is $2, 3$).
U_2 is homogeneous, thus is not inconsistent. It has more than one solution (it has not got enough equations to have exactly one solution).
U_3 is not homogeneous, and actually is inconsistent (equations 1. and 3. contradict each other).
U_4 is neither homogeneous, nor inconsistent, and has more than one solution.

2. Which tuple $(-1, 1, 1, -1), (2, 3, 1, 4), (4, -3, 2, 1), (4, 0, -3, \frac{1}{2})$ satisfy the following system of equations?

$$\begin{cases} 3x_1 + 2x_2 + 4x_3 + 2x_4 = 1 \\ 7x_1 + 5x_2 + 9x_3 + 4x_4 = 3 \\ 5x_1 - 3x_2 + 7x_3 + 4x_4 = 1 \end{cases}$$

$(-1, 1, 1, -1)$ does not satisfy equation 3., $(2, 3, 1, 4)$ does not satisfy equation 1., $(4, -3, 2, 1)$ does not satisfy equation 3, and the last tuple is a solution.

3. Find the general solution to the following system of equations.

$$\begin{cases} x_1 + 3x_2 + x_3 + 5x_4 = 2 \\ 2x_1 + 7x_2 + 9x_3 + 2x_4 = 4 \\ 4x_1 + 13x_2 + 11x_3 + 12x_4 = 8 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 2 \\ 2 & 7 & 9 & 2 & 4 \\ 4 & 13 & 11 & 12 & 8 \end{array}\right] \xrightarrow{w_2 - 2w_1, w_3 - 4w_1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 1 & 7 & -8 & 0 \end{array}\right] \xrightarrow{w_3 - 2w_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{w_1 - 3w_2} \left[\begin{array}{cccc|c} 1 & 0 & -20 & 29 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

thus the general solution is $\begin{cases} x_1 = 2 + 20x_3 - 29x_4 \\ x_2 = -7x_3 + 8x_4 \end{cases}$, and in parametric form: $(2 + 20x_3 - 29x_4, -7x_3 + 8x_4, x_3, x_4)$.

4. Find the general solution to the following system of equations.

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + 3x_5 = 2 \\ 6x_1 - 3x_2 + 2x_3 + 4x_4 + 5x_5 = 3 \\ 6x_1 - 3x_2 + 4x_3 + 8x_4 + 13x_5 = 9 \\ 4x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 1 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 2 & 3 & 2 \\ 6 & -3 & 2 & 4 & 5 & 3 \\ 6 & -3 & 4 & 8 & 13 & 9 \\ 4 & -2 & 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{w_2 - 3w_1, w_3 - 3w_1, w_4 - 2w_1} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 2 & 3 & 2 \\ 0 & 0 & -1 & -2 & -4 & -3 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & -1 & -3 & -4 & -3 \end{array}\right]$$

$$\xrightarrow{w_3 - 2w_2, w_4 - w_2} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 2 & 3 & 2 \\ 0 & 0 & -1 & -2 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_4} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 2 & 3 & 2 \\ 0 & 0 & -1 & -2 & -4 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{w_3 \cdot (-1), w_1 + w_2, w_2 \cdot (-1)} \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_2 - 2w_3, w_1 \cdot \frac{1}{2}} \left[\begin{array}{ccccc|c} 1 & \frac{-1}{2} & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

thus the general solution is: $\begin{cases} x_1 = \frac{-1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_5 \\ x_3 = 3 - 4x_5 \\ x_4 = 0 \end{cases}$, in parametric form we get: $(\frac{-1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_5, x_2, 3 - 4x_5, 0, x_5)$.

5. For which real numbers $t \in \mathbb{R}$ tuple $(1, t, 3, 2t)$ is a solution to the following system of equations?

$$\begin{cases} 3x_1 + 2x_2 + x_3 - x_4 = 6 \\ 2x_1 + 5x_2 - 3x_3 - 2x_4 = 5 \\ x_1 - 4x_2 + 5x_3 + 2x_4 = 16 \end{cases}$$

By substitution we get

$$\begin{cases} 3 + 2t + 3 - 2t = 6 \\ 2 + 5t - 9 - 4t = 5 \\ 1 - 4t + 15 + 4t = 16 \end{cases},$$

so

$$\begin{cases} 6 = 6 \\ t = 12 \\ 16 = 16 \end{cases},$$

thus this tuple is a solution if and only if $t = 12$.

6. For which real numbers $s \in \mathbb{R}$ the system of equations

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 3x_4 = 2 \\ 3x_1 + 5x_2 + 4x_3 + 8x_4 = 7 \\ x_1 + 3x_2 + 4x_3 + 4x_4 = s \end{cases}$$

is consistent?

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 2 \\ 3 & 5 & 4 & 8 & 7 \\ 1 & 3 & 4 & 4 & s \end{bmatrix} \xrightarrow{w_2 - 3w_1, w_3 - w_1} \begin{bmatrix} 1 & 2 & 2 & 3 & 2 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 1 & 2 & 1 & s - 2 \end{bmatrix} \xrightarrow{w_3 + w_2}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 2 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & s-1 \end{bmatrix}$$

Which means, that the system is not inconsistent if and only if $s = 1$.

7. Let $w(x)$ be a polynomial of degree 3 satisfying conditions $w(0) = -1, w(1) = 3, w(2) = 7, w(-1) = -5$. Find the coefficients of $w(x)$.

   Let $w(x) = ax^3 + bx^2 + cx + d$. From the first condition we get that $d = -1$. Other conditions can be written in the form of the following system of equations

$$\begin{cases} a+b+c=4 \\ 8a+4b+2c=8 \\ -a+b-c=-4 \end{cases},$$

   which can be easily solved, and we get $a = 0, b = 0, c = 4$, so we get a polynomial $4x - 1$ (which is actually of degree 1).

8. Three brothers Antoni, Bonifacy and Cezary added up their ages, and found out that they are in total 100 years old. Moreover, 10 years ago, the age of Antoni was equal to the sum of the age of Bonifacy and half of the age of Cezary. Is it possible, that the sum of the age of Antoni from 25 years ago multiplied by 4, and the age of Cezary now equals 100 years as well?

   We simply have to check whether the following system of equations is consistent:

$$\begin{cases} a+b+c=100 \\ 2a-2b-c=-10 \\ 4a+c=200 \end{cases}$$

   But it is inconsistent, hence it is not possible.