Linear algebra, WNE, 2018/2019
meeting 2. – homework solutions

4 October 2018

1. Find the general solution to the following system of equations.

$$\begin{cases} 6x + 4y + 5z + 2w + 3t = 1 \\ 3x + 2y + 4z + w + 2t = 3 \\ 3x + 2y - 2z + w = -7 \\ 9x + 6y + z + 3w + 2t = 2 \end{cases}.$$

We transform the matrix of this system to the reduced echelon form:

$$\left[\begin{array}{ccccc|c} 6 & 4 & 5 & 2 & 3 & 1 \\ 3 & 2 & 4 & 1 & 2 & 3 \\ 3 & 2 & -2 & 1 & 0 & -7 \\ 9 & 6 & 1 & 3 & 2 & 2 \end{array}\right] \xrightarrow{w_1 \leftrightarrow w_2} \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 6 & 4 & 5 & 2 & 3 & 1 \\ 3 & 2 & -2 & 1 & 0 & -7 \\ 9 & 6 & 1 & 3 & 2 & 2 \end{array}\right] \xrightarrow{w_2 - 2w_1, w_3 - w_1, w_4 - 3w_1}$$

$$\left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & -3 & 0 & -1 & -5 \\ 0 & 0 & -6 & 0 & -2 & -10 \\ 0 & 0 & -11 & 0 & -4 & -7 \end{array}\right] \xrightarrow{w_2 \cdot \frac{-1}{3}} \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & -6 & 0 & -2 & -10 \\ 0 & 0 & -11 & 0 & -4 & -7 \end{array}\right] \xrightarrow{w_3 - 6w_2, w_4 + 11w_2}$$

$$\left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{3} & \frac{34}{3} \end{array}\right] \xrightarrow{w_3 \leftrightarrow w_4} \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 & \frac{-1}{3} & \frac{34}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_3 \cdot (-3)}$$

$$\left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 1 & -34 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1 - 2w_3, w_2 - \frac{1}{3}w_3} \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 0 & 71 \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 & 1 & -34 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1 - 4w_2}$$

$$\left[\begin{array}{ccccc|c} 3 & 2 & 0 & 1 & 0 & 19 \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 & 1 & -34 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{w_1 \cdot \frac{1}{3}} \left[\begin{array}{ccccc|c} 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \frac{19}{3} \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 & 1 & -34 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the general solution is

$$\begin{cases} x_1 = \frac{19}{3} - \frac{2}{3}x_2 - \frac{1}{3}x_4 \\ x_3 = 13 \\ x_5 = -34 \end{cases}$$

In the parametric form: $(\frac{19}{3} - \frac{2}{3}x_2 - \frac{1}{3}x_4, x_2, 13, x_4, -34)$.

2. Does there exist a polynomial $w(x)$ of degree 2 such that $w(-2) = 2, w(3) = 10$ which has both real roots such that their product equals -3? *Hint: use Vieta's formulas.*

Assume that $w(x) = ax^2 + bx + c$. We have the following system of equations:

$$\begin{cases} 4a - 2b + c = 2 \\ 9a + 3b + c = 10 \\ 3a + c = 0 \end{cases}$$

Write the matrix of this system with column order c, a, b, and transform it into reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 2 \\ 1 & 9 & 3 & 10 \\ 1 & 3 & 0 & 0 \end{array}\right] \xrightarrow{w_2 - w_1, w_3 - w_1} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 2 \\ 0 & 5 & 5 & 8 \\ 0 & -1 & 2 & -2 \end{array}\right] \xrightarrow{w_2 \leftrightarrow w_3} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 0 & 5 & 5 & 8 \end{array}\right] \xrightarrow{w_3 + 5w_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 15 & -2 \end{array}\right] \xrightarrow{w_3 \cdot \frac{1}{15}, w_2 \cdot (-1)} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -\frac{2}{15} \end{array}\right] \xrightarrow{w_1 + 2w_3, w_2 + 2w_3}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & \frac{26}{15} \\ 0 & 1 & 0 & \frac{26}{15} \\ 0 & 0 & 1 & -\frac{2}{15} \end{array}\right] \xrightarrow{w_1 - 4w_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-26}{5} \\ 0 & 1 & 0 & \frac{26}{15} \\ 0 & 0 & 1 & -\frac{2}{15} \end{array}\right]$$

Therefore such a polynomial exists, and it is equal to $\frac{26}{15}x^2 - \frac{2}{15}x - \frac{26}{5}$.

3. For which real numbers $t \in \mathbb{R}$ tuple $(t^2, -1, 1, -t^2, 1)$ is a solution of the following system of equations.

$$\begin{cases} 7x_1 - 5x_2 - 3x_3 + 5x_4 - 5x_5 = -1 \\ 9x_1 + 8x_2 - 9x_3 + 2x_4 + 11x_5 = 1 \\ -4x_1 + 6x_2 + 2x_3 - x_4 + 9x_5 = 2 \end{cases}$$

By substitution we get $2t^2 = 2, 7t^2 = 7, -3t^2 = -3$, so the tuple is a solution for $t = \pm 1$.