Chromatic Variants of the Erdös-Szekeres Theorem on Points in Convex Position

[Extended Abstract]

Olivier Devillers
INRIA
BP93
06902 Sophia-Antipolis
France
Olivier.Devillers@sophia.inria.fr

Ferran Hurtado
Dept. Matemática Aplicada II,
Univ. Politécnica de Catalunya
Pau Gargallo 5
08028 Barcelona, Spain
hartado@ma2.upc.es

Carlos Seara
Dept. Matemática Aplicada II,
Univ. Politécnica de Catalunya
Pau Gargallo 5
08028 Barcelona, Spain
seara@ma2.upc.es

ABSTRACT

Let $S$ be a point set in the plane in general position, such that its elements are partitioned into $k$ classes or colors. In this paper we study several variants on problems related to the Erdös-Szekeres Theorem about subsets of $S$ in convex position, when additional chromatic constraints are considered.

1. INTRODUCTION AND PRELIMINARY RESULTS

The following result is commonly called the Erdös-Szekeres Theorem:

**Theorem 1.1.** [9] For every natural number $m$ there exists a number $n(m)$ such that any $n$-point set $S$ in the plane in general position with $n \geq n(m)$ contains an $m$-subset of points in convex position.

This problem has been attracting the attention of many researchers, both because its beauty and elementary statement, and because finding the exact value of $n(m)$ turns out to be a very challenging problem. The reader is referred to the survey paper [18] for a history of the problem, a description of many variants, and a wide list of references. The best currently known bounds are

$2^{n-2} \leq n(m) \leq \left(\frac{2m - 5}{m - 2}\right) + 2$,

where the lower bound was essentially proved by Erdös and Szekeres in their first papers and the upper bound is due to Tóth and Valtr [26].

Let $A$ be a point set in the plane in general position. An $m$-point subset $B \subset A$ in convex position is called an $m$-hole in $A$ if the convex hull $conv(B)$ is a polygon whose interior does not contain any point of $A$.

In 1978 Erdös [8] raised the following problem: is there a number $h(m)$, for every natural $m$, such that any $n$-point set $S$ in the plane in general position with $n \geq h(m)$ contains an $m$-hole?

Obviously $h(3) = 3$ and it is easy to see that $h(4) = 5$ and that $h(5) \geq 10$ (see Figure 1). Harborth [10] proved in 1978 that $h(5) = 10$, and in 1983 Horton [11] showed that $h(m)$ does not exist for $m \geq 7$ by constructing arbitrarily large sets without a 7-hole. The existence of $h(6)$ is a problem that still remains open.

The seminal paper by Erdös and Szekeres already mentioned the generalization of their original problem to higher dimensions, but even in the plane many variants have been considered, we mention next some examples. Bistriczy and Fejes Tóth [5] proved a generalization replacing points with convex bodies. Bárány et al [4], Caro [6] and Károlyi et al [15] gave results on the conjecture from [4] that for any given integers $m$ and $q$ any set large enough contains

![Figure 1: A set of 9 points with no 5-hole.](image-url)
a $n$-set for which the number of interior points is divisible by $q$. Several authors have studied the number of subsets in convex position of a given size that a sufficiently large point set can have [3, 25, 22, 23, 17, 2, 26, 7]. Let us finally mention the papers by Ambaramunjan [1], Károlyi [13], Hosono and Urabe [12] and Urabe [27], where several issues on partitioning a point set into subsets in convex position are considered.

Let $S = S_1 \cup \cdots \cup S_k$ be a partition of a planar point set $S$ in general position in the plane; we refer to $S_i$ as the set of points of color $i$. A subset $T \subseteq S$ is called monochromatic if all its points have the same color, and polychromatic otherwise. The term heterochromatic is used for the special case in which every element in $T$ has a different color.

In this paper we consider the following collection of problems. Given an integer $m$ and a set $S$ as above, possibly with additional requirements for $n = |S|$ to be large enough, can we find an $m$-hole of $S$ falling into one of the three described chromatic classes? Or an $m$-subset in convex position?

The original motivation for us to study these problems came from different areas. A finite set $\Gamma$ of curves in the plane is a separator for the sets $S_1, \ldots, S_k$ if every connected component in $\mathbb{R}^2 - \Gamma$ contains objects only from some $S_i$. We also say that each connected component is monochromatic. A thorough study of the subject is developed in [24].

When we have two sets, say the red points and the blue points, another way to approach their separability is to look for triangulations in which as many edges as possible (or as many triangles as possible) are monochromatic, which somehow contributes to isolate the two populations.

As a consequence of the above motivation, in this paper we study the conditions for the existence of some configuration, and also consider how many compatible such configurations can we guarantee, where compatibility stands for having disjoint relative interiors. For example, if the configuration is a monochromatic edge, we also try to find how many monochromatic edges can we guarantee without producing any crossing; the compatibility allows the edges to be completed to a triangulation.

In the sequel we will study the numbers $n_M(m, k)$, the minimal number of points colored with $k$ colors to ensure the existence of one monochromatic $m$-subset in convex position, and $MC(n, m, k)$, the minimal number of compatible monochromatic $m$-holes in a set of $n$ points colored with $k$ colors, $n_H(m, k)$, $n_P(m, k)$, $HC(n, m, k)$ and $PC(n, m, k)$ are similarly defined in the heterochromatic and polychromatic cases. A related (yet quite different) problem is considered in [19]. More in the spirit of our problems, several results are described in [20, 21, 14, 16] but looking for configurations like cycles or paths when edges are colored.

### 2. SUBSETS IN CONVEX POSITION

It is natural to start by considering subsets in convex position without the additional constraint of requiring empty interiors. While the former case is quite direct, the latter one, studied in the following section, is much simpler.

**Theorem 2.1.** $n_M(m, k) = k \cdot (n(m) - 1) + 1$.

**Theorem 2.2.**

If $k \geq n(m)$ then $n_H(m, k) = n(m)$.

If $k < n(m)$ then $n_H(m, k) = \infty$.

![Figure 2: A set of 18 points with no monochromatic 4-hole.](image)

The more interesting situation arises in the polychromatic case.

**Theorem 2.3.**

If $k \geq n(m)$ then $n_P(m, k) = n(m)$.

If $k < n(m - 1)$ then $n_P(m, k) = \infty$.

There is a gap remaining for $n(m - 1) \leq k < n(m)$. We have examples showing that $n_P(5, n(4)) = n_P(5, 5) = \infty$ and that $n_P(6, n(5)) = n_P(6, 9) = \infty$.

### 3. EMPTY CONVEX SUBSETS

#### 3.1 Monochromatic holes

**Theorem 3.1.** The minimum number of compatible monochromatic $m$-holes, $MC(n, m, k)$ are as follows:

<table>
<thead>
<tr>
<th>$m \setminus k$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The problem of finding $MC(n, 4, 2)$ remains open, however we can remark that a bichromatic set with no monochromatic 4-hole cannot contain any 7-hole, otherwise such heptagon would have at least 4 points of the same color and they would form a monochromatic 4-hole. As a consequence, any example showing $MC(n, 4, 2) = 0$ must be a Horton set or an equivalent construction.

A construction with 18 points which contains no monochromatic 4-hole is shown on Figure 2 proving that $MC(n, 4, 2) = 0$ for $n \leq 18$.

**Conjecture 3.1.** For $n$ large enough, $MC(n, 4, 2) > 0$. In other words, every large bichromatic point set contains some monochromatic 4-hole.
3.2 Heterochromatic holes

Theorem 3.2. The minimum numbers of compatible heterochromatic m-holes, \( MH(n, m, k) \) are the following:

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( n + k - 3 )</td>
<td>( n + k - 3 )</td>
<td>( n - 2 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( n + k - 3 )</td>
<td>( n - 2 )</td>
<td>( 0 ) ( (n \geq 2k - 3) )</td>
<td>( 0 ) ( (n \geq 2k - 4) )</td>
<td>( 0 ) ( (n \geq 2k - 5) )</td>
<td></td>
</tr>
</tbody>
</table>

For \( \left\lceil \frac{k(m-k+1)+1}{m-2} \right\rceil \leq n < 2k - m + 1 \) (with \( m \geq 4 \)) there is a gap in our results in which we do not know whether \( HC(n, m, k) \) is non-zero. \( HC(4, 4, 4) \) and \( HC(6, 4, 5) \) are shown to be zero by example.

3.3 Polychromatic holes

Theorem 3.3. The minimum numbers of compatible polychromatic m-holes, \( MP(n, m, k) \), are the following:

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( n + k - 3 )</td>
<td>( n + k - 3 )</td>
<td>( n - 2 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( n + k - 3 )</td>
<td>( n - 2 )</td>
<td>( 0 ) ( (n \geq 2k - 3) )</td>
<td>( 0 ) ( (n \geq 2k - 4) )</td>
<td>( 0 ) ( (n \geq 2k - 5) )</td>
<td></td>
</tr>
</tbody>
</table>

Notice that for \( m \geq 7 \) we can not expect polychromatic holes since there are sets with no 7-holes at all, and that even without colors the existence of 6-holes is yet an open problem.

4. CONCLUSION

Several results on a generalization of the Erdős-Szekeres Theorem to colored sets of points have been presented in this paper. The chromatic version with \( k \) colors differs significantly from the non-chromatic version since for fixed \( m \) and \( k \) and \( n \) large at will, it is possible to construct point sets with no heterochromatic or polychromatic subset of size \( m \) in convex position.

The more interesting results arise for the problem of the existence of \( m \)-holes for \( 3 \leq m \leq 6 \). We have succeeded in proving some results on the existence or non-existence of \( m \)-holes, but some intriguing problems remain open. Among them, our conjecture on the existence of a polychromatic 4-hole in any large enough bichromatic point set is maybe the most challenging one.

5. REFERENCES