Summary of professional accomplishments

1. **Name and surname:** Henryk Michalewski
2. **Diplomas:**
   - PhD in Mathematics defended in 2003 at the Faculty of Mathematics, Informatics and Mechanics of the University of Warsaw. The thesis title: *Function spaces with the topology of pointwise convergence.*
   - Master's degree in Mathematics defended in 1998 at the Faculty of Mathematics, Informatics and Mechanics of the University of Warsaw. The thesis title: *Function spaces and hereditary Baire spaces.*
3. **Academic employment:**
   - Since October 2004: assistant professor, Institute of Mathematics, University of Warsaw.
   - Since October 2002: teaching assistant, Institute of Mathematics, University of Warsaw.
   - On leave: winter semester 2002/2003 — Fields Institute, Toronto, a semester on applications of set theory, from October 2005 to September 2007 — postdoc at the Ben Gurion University in Israel, from October 2011 to September 2012 — Institute of Mathematics, Warsaw.
4. **Specific research accomplishment:** A series of 5 publications entitled

Investigations of automata and related logics using methods of set theory

Publications comprising the series:

(A) M. Bojańczyk, T. Gogacz, H. Michalewski and M. Skrzypczak, *On the decidability of MSO+U on infinite trees*, International Colloquium on Automata, Languages and Programming (ICALP), 50–61, Copenhagen 2014.

(B) T. Gogacz, H. Michalewski, M. Mio and M. Skrzypczak, *Measure properties of game languages*, conference Mathematical Foundations of Computer Science (MFCS), 303–314, Budapest 2014.

(C) S. Hummel, H. Michalewski, D. Niwiński, *On the Borel Inseparability of Game Tree Languages*, Symposium on Theoretical Aspects of Computer Science (STACS), 565–575, Freiburg 2009.

(D) H. Michalewski, D. Niwiński, *On topological completeness of regular tree languages*, Logic and Program Semantics 2012 - *Essays Dedicated to Dexter Kozen on the Occasion of His 60th Birthday*, 165–179.

(E) A. Arnold, H. Michalewski, D. Niwiński, *On the separation question for tree languages*, Theory of Computing Systems 55 (2014), 833-855. This is an extended version of the article [2], presented during the Symposium on Theoretical Aspects of Computer Science (STACS), 2012.

The series of publications is focused on set-theoretical properties of automata on trees. I started to be interested in this topic in 2008. My earlier work concerns set-theoretical properties of other mathematical structures such as function spaces, measures and real functions. The description below consists of the introduction, describing connections between the automata theory and set theory and three chapters describing content of the papers in the series.

Entscheidungsproblem

Introduction. In the beginning of the 20th century, along with axiomatization of mathematical theories, for a given formula φ written in the language of set theory, arithmetic or more generally in the first order logic, it became a realistic perspective to decide automatically whether φ can be deduced from given axioms. This research project, *Entscheidungsproblem* or in modern terminology *a classical decision problem*, was formulated by D. Hilbert in year 1928 in the book [27, page 8]][1]. This problem turned to be very inspiring for the development of logic, however the answers have not completely fulfilled the expectations of Hilbert. On one hand, Gödel's Completeness Theorem confirmed that every formula being a consequence of the axioms can be formally deduced from the axioms. On the other hand, Gödel's Incompleteness Theorem showed that most of mathematical theories (including arithmetic) cannot be completely axiomatized. Finally, A. Church and A. Turing ([16, 63]) discovered, that the deduction from the axioms cannot be turned in an automatic decision process; that is, in cases when the deduction does not exist, in general one cannot expect an automatic answer.

The above results tell us that the contemporary computers cannot solve the Riemann Hypothesis or find all errors in the Windows Operating System. Along with the negative results related to the Entscheidungsproblem, already in the 1930-ties there were shown some positive theorems. A. Tarski proved ([62]) that if we limit our attention to the theory of the reals with addition and multiplication, then there exists an algorithm which for a given sentence deduces a proof or says that the formula is false. M. Presburger ([52]) proved an analogous result about the theory of the natural numbers with addition. From the mathematical perspective both theories are not very rich, however one can express in them a correctness of computer programs which can be written in the language of linear programming. Despite the fact, that in full generality Entscheidungsproblem was solved in the negative, these two positive results show that decidable theories may have very serious practical impact.

[1] See also Section 12 in the 3rd Chapter of book [26] written in the same period, as well as modern source [11].

mathematical difficulties and extensions of decidable theories

I claim that along with discoveries of even stronger decidable theories, we are coming closer and closer to algorithmic problems which pose serious mathematical challenges. In particular, some of these issues can be analyzed using methods of set theory. This claim is illustrated by the evolution of decidable theories from the Presburger arithmetic to two monadic theories. The monadic theories proved to be decidable but as we show in the paper (A), they are rather hard to extend without loosing decidability. Before we move to this topic, I would like to recall some well known facts about the second order theories.

Taking into account that already the first order theory of the natural numbers with addition and multiplication is undecidable, clearly the second order theory of the natural numbers with addition and multiplication is undecidable. Indeed, the second order theory of natural numbers without any operations or relations is already undecidable, because one can define in it the addition and multiplication ([55]). Similarly, even if we restrict the second order quantification only to subsets, that is if we limit our attention to the monadic second order logic, then having addition, one can already define multiplication on the natural numbers ([55]), hence the monadic second order theory of the natural numbers with addition is undecidable.

Besides these two remarks, in the article of R. Robinson [55] are mentioned two questions posed by Tarski during a lecture:

Tarski's problems

1. *Is it possible to define in the monadic second order logic, the addition on the natural numbers only in terms of the successor operation?*[2]
2. *Is there an algorithm deciding the monadic second order theory of the natural numbers with the successor operation?*

Answers to these questions were given in 1960 by R. Büchi ([14]), who showed that the theory S1S, that is the monadic second order theory of the natural numbers with the successor operation is decidable. This gives a positive answer to the second Problem and a negative answer to the first Problem. Büchi reduced the problem of satisfiability of a given formula φ to the question about existence of a run of a certain automaton $\mathcal{A}_\varphi$ on infinite words (see Figure 1). The last question can easily be decided through inspection of the structure of the automaton $\mathcal{A}_\varphi$.

Büchi's theorem and its consequences

As in the case of the theory of the real numbers with addition and multiplication and the Presburger arithmetic, the decidability of the theory S1S implies interesting consequences for the verification of some special programs. At the beginning it seemed to be an interesting curiosity, however over time this method was adapted to the verification of logical circuits and subsets of bigger programs and currently is applied on the industrial scale. The idea of the verification is very simple: as far as the desired property of the circuit is expressible in S1S or another theory decidable thanks to the Büchi algorithm, a computer can automatically establish if the property of the circuit holds and if the answer is negative, then the

[2] Using the successor operation one can encode the Presburger arithmetic, but this problem is not about the encoding of arithmetic, but about the expressibility of the relation $\{(x, y, z) : x + y = z\}$.

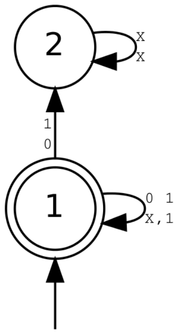

Figure 1: The automaton A_φ corresponding to the formula $\varphi(X, Y) = X \subseteq Y$. The alphabet consists of the symbols $\{\genfrac{}{}{0pt}{}{0}{0}, \genfrac{}{}{0pt}{}{1}{0}, \genfrac{}{}{0pt}{}{0}{1}, \genfrac{}{}{0pt}{}{1}{1}\}$, the initial and accepting state is 1, the transition $\genfrac{}{}{0pt}{}{0}{X}$ means that on the first coordinate 0 is expected and there are no restrictions on the second coordinate, the transition $\genfrac{}{}{0pt}{}{X}{X}$ means that arbitrary numbers are accepted on both coordinates.

computer may generate a counterexample. Obviously, practical uses are limited by the availability of programming tools convenient for the engineers and logics used in practice, such as LTL , may have considerable different syntax than the monadic logic.

automata on infinite words

The above mentioned automaton $\mathcal{A}$ on infinite words consists of a finite alphabet A, a finite set of states $Q_\mathcal{A}$ along with a subset of accepting states $F_\mathcal{A} \subset Q_\mathcal{A}$, an initial state $q_0 \in Q_\mathcal{A}$ and a set of transitions $\Delta_\mathcal{A} \subseteq Q_\mathcal{A} \times A \times Q_\mathcal{A}$. The automaton $\mathcal{A}$ one can consider as a finite graph, with the set $Q_\mathcal{A}$ as the set of vertices. Elements of the set $\Delta_\mathcal{A}$ can be interpreted as edges of the graph with labels from the set A. An infinite word $w \in A^\omega$ is accepted by the automaton $\mathcal{A}$ if there exists a sequence of transitions $\delta_0, \delta_1, \ldots \in \Delta_\mathcal{A}$, such that $\delta_0 = (q_0, w_0, q_1)$, $\delta_1 = (q_1, w_1, q_2)$ and so on. The accepting condition requires, that for infinitely many $n \in \omega$, the state q_n belongs to the set of accepting states $F_\mathcal{A}$.

Rabin's theorem

The result of Büchi was extended by M. O. Rabin ([53]), who proved that the monadic theory S2S of the binary tree with two successors is decidable, reducing the question of satisfiability of a given formula φ to the question whether there exists an accepting run of the automaton $\mathcal{A}_\varphi$ on infinite trees. Decidability of this stronger theory allows an automatic verification of a broader spectrum of programs, than in the case of the theory S1S.

games on countable boards

Despite the fact, that Rabin's argument is similar to the one used by Büchi, details are different, including a use of transfinite induction in the proof of the correctness of Rabin's algorithm. Y. Gurevich and L. Harrington ([24]) proved, that one can replace the transfinite induction with a finite–memory determinacy of a certain game on a countable board, where the winning condition is expressed as a Borel set of low complexity.

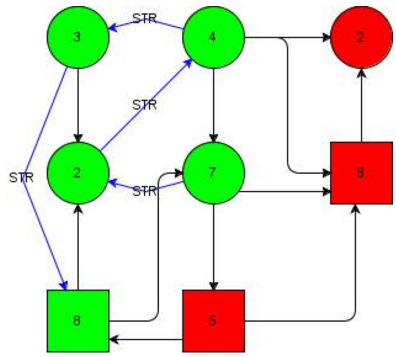

Figure 2: A parity game with 8 states; Eve decides about the moves from states marked with circles, Adam decides about the moves marked with squares. A positional winning strategy of Eve is marked with edges labeled "STR". From the green states Eve has a winning strategy, from other states Adam has a winning strategy.

More precisely, in the proof of Rabin's theorem one can apply a parity game[3], where the set of states V is divided into states $V_\exists$ belonging to Eve and states $V_\forall$ belonging to Adam. To every state $q \in V$ there is assigned a rank $\mathsf{rank}(q) \in \omega$. Possible moves from a given state are marked by edges between vertices V. Eve wins an infinite game $q_0, q_1, \ldots$ if

$$\limsup_{n \in \omega} \mathsf{rank}(q_n)$$

is an even number. Figure 2 shows an example of such a game.

nondeterministic and alternating automata on infinite trees

Nondeterministic automata on trees, making appearance in the proof of Rabin's theorem, have similar structure to the automata on infinite words, however the transitions of the automaton take into consideration two successors of a given state, hence the set of all transitions $\Delta_\mathcal{A}$ is a subset of the set $Q_\mathcal{A} \times A \times Q_\mathcal{A} \times Q_\mathcal{A}$. Instead of defining a set of accepting states, the automaton is equipped with a function $\mathsf{rank} : Q_\mathcal{A} \to \omega$. A tree t, understood as a mapping $t : 2^* \to A$, is accepted by the automaton $\mathcal{A}$, if there exist an accepting run ρ of the automaton $\mathcal{A}$ on the tree t, that is a mapping $\rho : 2^* \to \Delta_\mathcal{A}$ such, that

1. labels of the tree t agrees with the transitions indicated by the run ρ,
2. for every infinite path π in 2^*, if $q_0, q_1, \ldots$ are the states reached on the path π during execution of the run ρ, then

   $$\limsup_{n \in \omega} \mathsf{rank}(q_n)$$

   is an even number.

[3]Gurevich and Harrington in [24] used more complicated games, parity games were introduced later in [19].

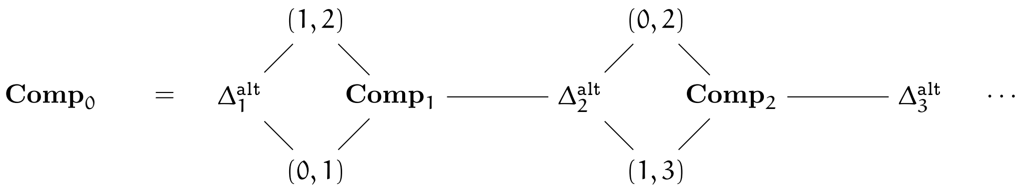

Figure 3: Mostowski–Rabin index hierarchy. For a given index (i,k), the dual index means the index above/below (i,k), for example $(1,3)$ and $(2,0)$ are dual. The classes Δ_k^{alt} are intersections of dual classes and the classes $\mathbf{Comp}_k$ consists of languages obtained through composition of automata of lower index for $k > 0$ and all weak alternating languages for $k = 0$.

An alternating automaton $\mathcal{A}$ on trees is even more closely related to the parity games. The set of states $Q_\mathcal{A}$ is divided into states belonging to Eve ($Q_\mathcal{A}^{\exists}$) and the states belonging to Adam ($Q_\mathcal{A}^{\forall}$). Transitions $\Delta_\mathcal{A} \subseteq Q_\mathcal{A} \times Q_\mathcal{A} \times (\{A \cup \epsilon\}) \times \{0,1\}$ one can identify with the edges of a directed graph with vertices $Q_\mathcal{A}$ and edges labeled by the set $(A \cup \{\epsilon\}) \times \{0,1\}$, interpreted as letters from the set $A \cup \{\epsilon\}$, along with the direction left or right in the tree. The automaton $\mathcal{A}$ is equipped with a mapping $\mathsf{rank} : Q_\mathcal{A} \to \omega$. One can play infinite parity game on the tree t starting from the initial state q_0. We say that t is accepted by $\mathcal{A}$, if there exists a winning strategy for Eve on the tree t.

We say that a language L is of index (i,k) for $i = 0,1$ and $k \in \omega$ $(i < k)$, if there exists $\mathcal{A}$ an alternating automaton on infinite trees accepting language L and such that the mapping rank takes values in the closed interval $[i,k] \subset \omega$ (see Figure 3).

J. Bradfield ([12]) proved that the hierarchy is strict, that is for every $i = 0,1$, $k \in \omega$, $i < k$ there exists a language L of index (i,k), which is not of the dual index.

A difficult step in the proof of Rabin's theorem consists in proving, that the complement of a language accepted by a nondeterministic automaton on trees is accepted by another nondeterministic automaton on trees. On the other hand, alternating automata are clearly closed with respect to the complementation, because it is enough to exchange the role of the players and move the values of the mapping rank by 1 (that is, the complementation can be expressed as the switching to the dual class). So, one can think about the proof of Rabin's theorem as a transformation of an alternating automaton on trees into a nondeterministic automaton on trees. In particular, E. A. Emerson and C. S. Jutla expressed ([19], see also [48]) Rabin's theorem as equality between the classes of languages defined by nondeterministic and alternating automata on trees.

determinization of automata on trees

The theory of automata on infinite trees seems to be substantially more complicated than the theory of automata on infinite words. One of the reasons is the fact, that automata on infinite words can be determinized ([41]) and this is false for automata on trees. In the case of automata on infinite tress, the

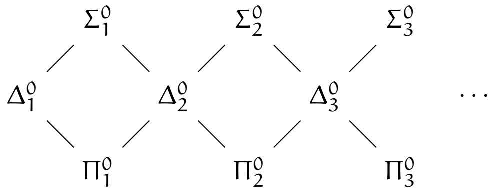

Figure 4: Borel hierarchy.

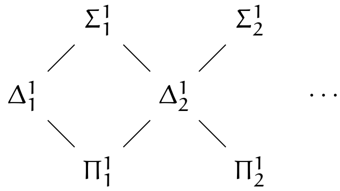

Figure 5: Projective hierarchy.

languages accepted by deterministic automata are significantly simpler, than the languages accepted by nondeterministic automata.

In terms of Mostowski–Rabin index hierarchy, the deterministic automata are of index $(0, 1)$. On the other hand, a characteristic feature of this class of automata is expressible in terms of descriptive set theory. The descriptive set theory allows to classify sets according to their topological complexity. The set A^{2^*} of all infinite trees over the alphabet A, is equipped with the product topology. Taking into account that the set A is finite, the resulting topology is compact. The simplest set in the descriptive hierarchy are clopen sets (Δ^0_1). To the family Δ^0_1 belong in particular sets

$$U_{v,a} = \{t \in A^{2^*} : t(v) = a\}$$

for $v \in 2^*$ and $a \in A$ and other elements of Δ^0_1 are Boolean combinations of the above sets. Countable unions and intersections produce next levels of the Borel hierarchy, that is the open sets (Σ^0_1), closed sets (Π^0_1), F_σ (Σ^0_2), G_δ (Π^0_2) and so on.

Borel hierarchy and projective hierarchy

Borel sets are denoted Δ^1_1 and their continuous images are called analytic sets (Σ^1_1). The complements of analytic sets are called coanalytic sets (Π^1_1). Continuous images of coanalytic sets are Σ^1_2 sets and their complements are Π^1_2 sets. The languages accepted by deterministic automata on infinite trees are coanalytic and the languages accepted by nondeterministic automata on infinite trees reach much further in the projective hierarchy (see Figure 5). From Rabin's theorem one can deduce, that the languages accepted by nondeterministic automata on infinite trees belong to the class Δ^1_2, that is they are simultaneously in the classes Σ^1_2 and Π^1_2.

Detailed analysis of descriptive complexity of these languages is given in the paper (B) — this is a corollary of our studies of the probabilistic μ-calculus. In the paper (B) we prove, that the languages accepted by nondeterministic automata on infinite trees are included in the first ω levels of Kolmogorov's hierarchy of $\mathcal{R}$-sets and one can find regular languages of infinite trees complete for each of the first ω levels of this hierarchy.

problems related to automata on infinite trees

Experience with automata on words suggests, that each algorithmic problem related to nondeterministic or alternating automata should be possible to settle analyzing the structure of transitions of a given automaton. It appears however, that many natural problems are combinatorially quite complicated and currently beyond the reach of known algorithms. In particular, so far there are no algorithms solving the following decision problems

- given a nondeterministic automaton on infinite trees, decide whether the language defined by this automaton is Borel,
- for a given index (i, k) and a nondeterministic automaton on infinite trees, decide whether the language defined by the automaton belongs to the (i, k)–th level of the Mostowski–Rabin index hierarchy.

Despite significant efforts (see in particular [21, 49, 51, 64]), without a big exaggeration one can say that essentially no nontrivial question regarding nondeterministic automata and the Mostowski–Rabin index hierarchy, Borel hierarchy or projective hierarchy, has an algorithmic solution. One exception is an algebraic characterization of the family of Boolean combinations of open sets ([9]), with a slight modification proposed by A. Facchini and myself in [20], where we give an algorithmic criterion for the membership in the family Δ^0_2. Regarding the index hierarchy, there are only partial results about the membership in the levels $(0, 1)$ and $(1, 2)$ ([17]).

In the series of papers ***Investigations of automata and related logics using methods of set theory***, the methods borrowed from the foundations of mathematics I consider to be a substitute of a proper algorithmic solution.

set theoretical description as a substitute of an algorithm

As shown in the papers included in the series, in some instances we obtain an interesting information about a hierarchy without a complete understanding of the combinatorial structure of the automata inducing the hierarchy. In particular, we can prove some theorems about separation property of regular languages (see Figure 6). More precisely, in some concrete cases we can answer whether languages A, B can be separated by a set belonging to a given simpler family of languages. The separation property is related to the problem of deciding the membership in a given level of a hierarchy. Suppose, that we have an algorithm which for given languages A, B can answer a question whether there exists a set from a family $\mathcal{C}$ separating A and B. This algorithm can be applied to A and its complement. The answer would be positive if and only if the set itself would be a member of $\mathcal{C}$. In the case of the Mostowski–Rabin index hierarchy, the separation property can be investigated with respect to the family $\mathbf{Comp}_k$ and a bigger family Δ^{alt}_k for $k \in \omega$. Providing an algorithm deciding the separation property of $\mathbf{Comp}_k$ or Δ^{alt}_k $(k \in \omega)$ seems to be out of reach of the currently available

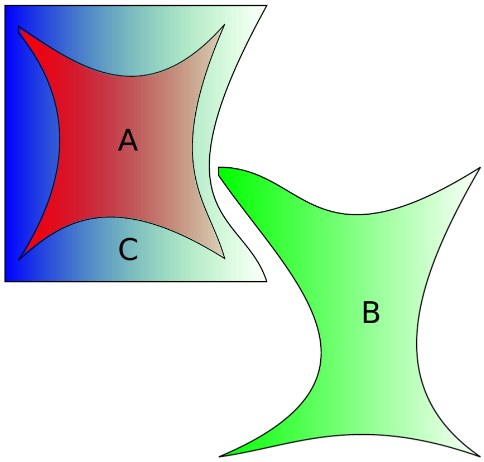

Figure 6: Set A can be separated from B by a simpler set C.

methods in automata theory. We approximate a hypothetical algorithm for the separation property with respect to the family Δ_k^{alt} by proving, that for every disjoint pair of languages of index $(1,2)$ the answer would be "YES" (for $k=1$) and providing, for every $k \in \omega$, combinatorially non-trivial examples of disjoint pairs, such that the answers would be "NO".

The separation property is well understood with respect to the Borel hierarchy ([31, Theorem 22.16]). The separation property of analytic sets was proved by N. Lusin ([31, Theorem 14.7]): two disjoint analytic sets can be separated by a Borel set. Papers (C),(D), (E) are attempting to translate these results from descriptive set theory to the realm of the Mostowski–Rabin index hierarchy (see a detailed description below).

methods of set theory as heuristics

I claim that set-theoretical methods play roles of heuristics in this series of papers.[4] A set-theoretical heuristic helps to get an intuition how to deal with a given problem and very likely it may no longer be needed, once a full understanding of the problem is reached. In the following three Sections I describe the paper (A), next the paper (B) and finally the three papers (C),(D) and (E). Set-theoretical heuristics play roles in all of these research directions.

Description of the paper (A). This is a part of a broader program of extending expressive power of MSO logic by additional quantifiers, which allow for a

[4]Regarding axioms considered new at that time, such as Continuum Hypothesis, K. Gödel writes in [22, page 261] about *verifiable consequences* as a criterion of a success of a given axiom. By *verifiable consequences* Gödel means theorems proved with some help of a given axiom, which can be proved without use of the axiom, at the expense of a more complicated proof. J. P. Burgess in [32] described this kind of success as a heuristic intuition in favor of an axiom and recalls the most famous example of such a success, namely Martin's Borel determinacy theorem, proved earlier with a help of a large cardinal assumption.

restricted counting. One choice for such an extension is the quantifier U added to theories S1S or S2S

$$U.\varphi(X) \equiv \forall_{n\in\omega} \exists_{X,|X|>n} \varphi(X).$$

A series of papers is dedicated to this topic: in [6] a plan of investigations of the quantifier U was formulated, connections with automata with counters were shown in [8], various fragments of logic MSO+U on words and trees were proved to be decidable [7, 10] and moreover in the paper [18] authors showed a reduction of decidability of the nondeterministic index problem do the decidability of S2S extended by the quantifier U[5].

In the paper (A) we prove that the monadic theory of the binary tree with two successors and the quantifier U is undecidable. This result was proved using an additional set–theoretic assumption V=L ([30, Chapter 13]).

Paper (A) is based on a method developed by Shelah in [61]. Shelah proved that the monadic theory of the reals is undecidable. Apart of the above mentioned new result about the quantifier U, we included a slight modification of Shelah's argument. Our proof is based on a reduction of the fragment $\forall^*\exists^*$ of the first order theory of undirected graphs without equality to the monadic theory of the reals. This fragment of the theory of graphs can be quite conveniently expressed using the tools provided by Shelah[6]. The fragment $\forall^*\exists^*$ of the first order theory of undirected graphs without equality is undecidable as shown by Y. Gurevich in [23, Theorem 1, Section 9], comp. [11, Chapter 3.2].

Shelah's theorem was proved in [61] under an additional assumption of the Continuum Hypothesis and the argument is based on the transfinite induction. In the paper of Y. Gurevich and S. Shelah [25] the transfinite induction from [61] was refined to the point, where the assumption of the Continuum Hypothesis was no longer required, at the cost of a more complicated combinatorial argument. The Continuum Hypothesis played a role of a heuristic, which at the initial stage allowed some combinatorial simplifications and finally was completely removed.

Continuum Hypothesis as heuristic

In (A) we use a stronger heuristic, than one used in [61]. Namely, the axiom V=L implies Continuum Hypothesis and allows to define a well–ordering of the reals, which belongs to the family Δ^1_2. This implies, that many subsets of the reals build through transfinite induction can be represented as projective sets. In particular, under the assumption V=L one can prove that among Δ^1_2 sets are Lebesgue non–measurable sets (this result was proved by K. Gödel, modern proof can be found in [30, Corollary 25.28]).

Axiom V=L as heuristic

Papers [28] and [29] showed, that in the logic MSO+U on infinite words, for every $n = 1, 2, \ldots$, one can define a set at the n–th level of the projective hierarchy and complete for all projective sets from this level with respect to continuous reductions.

We prove, that continuous reductions can be expressed through quantification over sets of nodes of an infinite tree (Lemma 2.5 in (A)), and then we rewrite the

[5]This is a corollary from [18] noticed by M. Bojańczyk.

[6]Proof in the paper [61] is based on the reduction of the first order theory of the natural numbers with addition and multiplication.

important sentences expressed in the monadic theory of the reals in terms of the monadic theory of the binary tree with two successors and the quantifier U. This implies undecidability of the later theory.

Description of the paper (B). The μ-calculus is a logic which, under appropriate interpretation ([19, 50]), is equivalent in terms of expressive power to alternating automata on infinite trees.

The μ-calculus along with weaker logics LTL and CTL, can express certain decidable properties of labeled transition systems. From the point of view of mathematical modeling, these are the properties expressible in terms of parity games. For example, given three types of states V_0, V_1, V_2 in the μ-calculus we may require, that after every visit in a state from V_0, a state from V_1 will be also visited and finally, a state from V_2 will be reached.

The probabilistic μ-calculus allows formulation of similar properties of a given model, for example: probability of reaching a state from V_1 after a visit in a state from V_0 is greater or equal than $\frac{1}{2}$, probability of reaching a state from V_2 and later remaining in states from V_2 is smaller or equal than $\frac{1}{4}$. Informally, one can say that probability plays a role of a new quantifier in a similar way as the quantifier U in the context of the MSO logic.

The logic PCTL is used in practice, but its theoretical properties are not yet completely understood, in particular the satisfiability problem for this logic is not settled ([13]). One of the motivations to study the probabilistic μ-calculus is to provide a more abstract framework, better suited for mathematical analysis of problems such as satisfiability of the PCTL logic. In general, a witness of satisfiability in the μ-calculus is an infinite tree (unfolding of a finite graph to a tree, according to the relations of the model) and in the probabilistic μ-calculus this is a set of trees; in the later case we are interested in the measure of this set. The probabilistic μ-calculus proposed by M. Mio allows an interpretation of the simple and practical logic PCTL and at the same time it has an expressive power analogous to the ordinary μ-calculus ([45, 46]). The correctness of Mio's approach is based on two premises:

problems related to measurability

1. **regular languages of trees are universally measurable**, that is measurable with respect to every Borel measure and moreover
2. **every Borel measure is continuous in appropriate sense on every regular language of trees.**

This is not a particularity of Mio's approach: measurability is a standard problem in modeling with the logic PCTL ([5, Lemma 10.39]), however so far the problem did not imply serious difficulties, because in measure-theoretic analysis of the logic PCTL make appearance only Borel sets, which are obviously universally measurable.

Both problems were partially solved by Mio with help of a set-theoretical heuristic guaranteeing that sufficiently simple sets in the projective hierarchy have the above two properties. The role of the heuristic was played by the

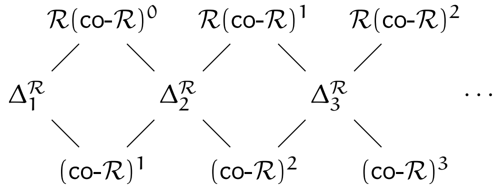

Figure 7: The hierarchy of $\mathcal{R}$–sets is obtained through iteration of the transformation co-$\mathcal{R}$ starting from the operation $\bigcup \circ \bigcap$; the transformation co- is a set–theoretical equivalent of the dualization of parity games, that is the roles of players are exchanged and the index is moved by 1. The operation $\mathcal{R}(\bigcup \circ \bigcap) = \mathcal{R}(\text{co-}\mathcal{R})^0(\bigcup \circ \bigcap)$ is the Suslin operation and generates the family of analytic sets, co-$\mathcal{R}(\bigcup \circ \bigcap)$ generates coanalytic sets, $\Delta_1^{\mathcal{R}}$ consists of Borel sets and $(\text{co-}\mathcal{R})^2(\bigcup \circ \bigcap)$ generates Σ_1^1–inductive sets, treated in more details in the paper (D).

Martin Axiom as heuristic

Martin Axiom ([30, Chapter 16]). In the paper (B) we show that no heuristic is required in order to prove the universal measurability, since the regular languages of infinite trees are located on the first ω levels of the Kolmogorov hierarchy of $\mathcal{R}$–sets. Our result is more precise: among the regular languages there are complete languages for the first ω levels of the hierarchy of $\mathcal{R}$-sets. According to our knowledge, these are in fact the first *natural* examples of $\mathcal{R}$-sets above the second level of this hierarchy. This extends an earlier result of J. Saint Raymond ([58]). In turn, the results of [58] were earlier applied in the paper (D). The second of the above problems posed by Mio is settled in (B) using a weaker heuristic than the Martin Axiom, however already after the paper was submitted for the publication, we discovered that a method developed by N. Lusin and W. Sierpiński in [39] shows that the continuity can be proved without any set–theoretical assumptions.

The original idea of Kolmogorov [35, Addendum 2] was to create a large σ–field consisting of universally measurable sets. The definition proposed by Kolmogorov is an extension of a definition of an analytic set. Kolmogorov's idea can be considered quite revolutionary even from the modern perspective. Namely, Kolmogorov creates inductively more and more complicated operations on sets. The first of the operations is $\bigcap \circ \bigcup$, after the first transformation we obtain the Souslin operation which is in correspondence with the index $(1, 2)$ of the Mostowski–Rabin hierarchy. The next transformation creates from the Souslin operation an operation corresponding to the index $(0, 2)$ and so on (see Figure 7).

The relation between $\mathcal{R}$-sets and Gale–Stewart games was for the first time noticed by J. P. Burgess in [15]. From this interesting theorem one can relatively straightforwardly deduce an answer to the first question posed by Mio, that is to the question about universal measurability. We decided to partially reconstruct Burgess' reasoning, what seems to be inevitable if one plans to prove more precise results about location of regular languages within the hierarchy of $\mathcal{R}$-sets.

As the whole theory of $\mathcal{R}$-sets, Burgess' method as described in [15] is hampered by notational problems and constant attempts to encode with natural numbers a certain data structure, which does not lend itself to such an encoding. In practice it makes proofs hard to write or read and in particular the game-theoretic characterization proved by Burgess is only written for one specific level of the $\mathcal{R}$-hierarchy. One of the decisions made in the paper (B) is to preserve the natural data structure related to the transformation $\mathcal{R}$. The operations $\bigcup$ and $\bigcap$ are defined for countable sequences $\langle A_n : n \in \omega \rangle$. The operation $\bigcup \circ \bigcap$ is defined for sequences index $\langle A_{n,m} : n, m \in \omega \rangle$ index by two variables $n, m \in \omega$. If a given operation φ as a domain has a sequence $\langle A_w : w \in \mathbb{A} \rangle$ indexed by a certain set $\mathbb{A}$, then after transformation, the operation $\mathcal{R}(\varphi)$ is defined over $\langle A_w : w \in \mathbb{A}^* \rangle$. For example, if φ operated on sequences indexed by natural numbers, then $\mathcal{R}(\varphi)$ will be defined on families of sets indexed by finite sequences of natural numbers. Two applications of transformation $\mathcal{R}$ will create an operation with a domain indexed by $(\omega^*)^*$ and so on. This is a familiar date structure called a *nested list*. In the paper we prove that there exists a natural correspondence between transformation $\mathcal{R}$ and generalized parity games, called in the paper the *matryoshka games*. The matryoshka games substitute the Gale–Stewart games used by Burgess. The natural analogy with the parity games results from correspondence between applications of the transformation $\mathcal{R}$ and increases of the index in parity games. The name *matryoshka games* is motivated by a three steps construction shown at Figures 8, 9 and 10. matryoshka games

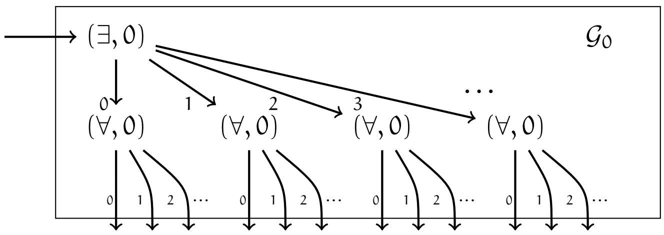

Figure 8: Game $\mathcal{G}_0$ consists of two rounds, a round of player $\exists$ and a round of player $\forall$; this is a game-theoretic counterpart of the operation $\bigcup \circ \bigcap$.

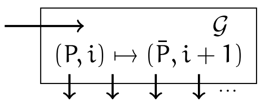

Figure 9: Transformation co- applied to a game $\mathcal{G}$.

The three transformations allow building more and more complicated matryoshka games, which correspond to higher and higher levels in the Mostowski-Rabin hierarchy. Sets

$$W_{i,k} = \{t : 2^* \to \{\exists, \forall\} \times \{i, \dots, k\} : \text{ player } \exists \text{ has a winning strategy on } t\}$$

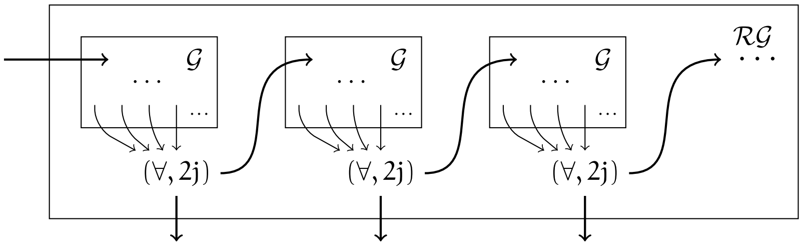

Figure 10: Transformation $\mathcal{R}$ applied to a game $\mathcal{G}$. The given game $\mathcal{G}$ is played as one of the stages of a larger game $\mathcal{RG}$.

are complete for the k–th level of the Kolmogorov hierarchy and the result is obtained through a characterization of transformation co-$\mathcal{R}$ in terms of matryoshka games.

The "low-level" set-theoretical definition of $\mathcal{R}$-sets has an advantage, that one can easily check basic properties, such as universal measurability. To be more precise, as Kolmogorov noticed in his notes [35, Addendum 2], without special difficulties one can apply the proofs known before for analytic sets[7]. On the other hand, the "high-level" characterization in terms of games allows for applications of $\mathcal{R}$-sets in the context of nondeterministic automata on infinite trees and the probabilistic μ-calculus.

Description of the papers (C),(D),(E). In an ideal scenario, for a given k we would like to have an algorithm, such that for given languages L_1, L_2 defined by automata on infinite trees, we would get an answer whether L_1, L_2 can be separated by a set from Δ_k^{alt}. At the moment this kind of algorithm is beyond reach, since we know too little about the combinatorial structure of the automata. Instead of solving this difficult algorithmic problem, in the above three papers we consider a somewhat simpler task of identifying levels of Mostowski–Rabin hierarchy, such that the separation property holds with respect to the family Δ_k^{alt} (comp. Figure 3).

In descriptive set theory, one can say, that the separation property is completely understood for the Borel hierarchy and for analytic sets. In the case of automata the understanding is only partial. Historically the first observation can be deduced from a proof of Rabin [54]: two disjoint languages of index $(1, 2)$ can be separated be a weak language, that is a language accepted by an alternating automaton such that the accepting condition is not expressed as lim sup but as maximum. An alternating automaton of index $(1, 2)$ can be simulated by a nondeterministic automaton of index $(1, 2)$[8] and in fact the proof, that two disjoint languages of index $(1, 2)$ can be separated by a weak language, can be deduced from this observation.

Rabin's result regarding index $(1, 2)$

[7] This rule does not apply to more subtle properties of analytic sets, such as the boundedness principle [30, Corollary 25.15].

[8] This is the only level of the Mostowski–Rabin index hierarchy where such a simulation is possible without increasing of the index.

A. Arnold and L. Santocanale proved in [4] that for even k, $k > 0$, a given two disjoint languages L_1, L_2 accepted by a nondeterministic automaton of index (i, k), can be separated by a language from the family $\mathbf{Comp}_{k-1}$. At the same time, it was proved in [4], that for $k > 0$ there exist languages in Δ^{alt}_{k+1} which do not belong to $\mathbf{Comp}_k$. In particular, every such language K has the property, that K and its compliment $\overline{K}$ have index $(0, k+1)$ and cannot be separated by a set from $\mathbf{Comp}_k$. These results of Rabin and Arnold–Santocanale suggest a conjecture, that for even k the separation property holds for the languages of index (i, k). Taking into account that in a typical hierarchy in descriptive set theory (comp. [31, Proposition 22.15]) the separation property cannot hold simultaneously for a given class and for the dual class, the above conjecture naturally extends to a full conjecture, that the separation property holds for index (i, k) if and only if k is even.

conjecture regarding separation property of index hierarchy

In the paper (C) we proved that there exists languages $L_{0,1}$, $L'_{0,1}$ of index $(0, 1)$ which cannot be separated by a Borel set, hence clearly they cannot be separated by a set from Δ^{alt}_1. This result is related to my earlier research in [36], where we also dealt with coanalytic sets inseparable by Borel sets. The proof in the paper (C) is based on the reduction of all disjoint pairs of Borel languages to the pair $(L_{0,1}, L'_{0,1})$. The existence of such a reduction shows that $L_{0,1}$ and $L'_{0,1}$ cannot be separated by a Borel set and consequently by a set from Δ^{alt}_1.

results of the paper (C)

In the paper (D), similarly as in the paper (C), we are interested in the topological complexity of the sets $W_{i,k}$, but the method used in (C) was substantially extended thanks to results from the paper of J. Saint Raymond ([58]) regarding Σ^1_1–inductive sets. We proved, that $W_{1,3}$ is a Σ^1_1–inductive set and complete for this class of sets. This result is interesting for the following reasons.

results of the paper (D)

- For the first time in the literature we precisely located the descriptive complexity of the set $W_{1,3}$ and all other languages of index $(1, 3)$. From the perspective of descriptive set theory, the class of Σ^1_1–inductive sets is understood only partially. There are known characterizations in terms of games, but on other hand the first *natural* example of a complete set was presented only few years ago in the paper [58] (J. Saint Raymond points to the discovery of this example as the main motivation behind the paper [58]).
- The result of the paper (D) helped to preliminary veryify the inseparability of the pair $(L_{1,3}, L'_{1,3})$[9] constructed in the paper (E), hence the set–theoretical methods played here a role of a heuristic. In the final version of the paper we managed to completely eliminate this heuristic in favor of a simpler fixed-point method, used in the context of $W_{i,k}$ languages in the papers [1, 3]. A similar proof technic as in the paper (D) was later applied in a more refined version in the paper (B) in order to prove, that languages $W_{i,k}$ are complete for the appropriate levels of the $\mathcal{R}$–hierarchy.

In the paper (E) we gave new examples of inseparable pairs for index (i, k), k odd. In the journal version we additionally showed, using the fixed-point method,

results of the paper (E)

[9] We proved that the pair $(L_{1,3}, L'_{1,3})$ is complete with respect to all disjoint pairs of Σ^1_1–inductive sets and in this context completeness implies inseparability.

that the separation property does not hold for the weak index hierarchy for levels $(1, k)$ $(k \in \omega, k > 1)$ and moreover, that the reduction property ([31, Definition 22.14]) fails for all levels of the index hierarchy. In the context of the descriptive set theory, the properties of reduction and separation are dual, as illustrated by the following examples:

1. F_σ sets have the reduction property, but do not have the separation property, their complements, that is G_δ sets, have the separation property, but do not have the reduction property.
2. analytic sets have the separation property, but do not have the reduction property, their complements, that is the coanalytic sets, have the reduction property but do not have the separation property.

In the Mostowski–Rabin index hierarchy this rule fails and as a result, one cannot reason as it is common in descriptive set theory ([31, Proposition 22.15(i)]), where the reduction property implies the separation property for the dual class. It seems that the separation property can be related to the product operation, which is defined for nondeterministic automata, but an analogous construction for the alternating automata is not known. For the meantime, the separation problem for the indices (i, k), k even, $k > 0$, remains open in all cases, except for the index $(1, 2)$. Also the separation problem is open for the weak hierarchy for all indices $(0, k)$, except for the index $(0, 1)$. In particular, it is unknown, whether two disjoint languages accepted by automata of index $(0, 2)$ can be separated by a language, which is simultaneously of index $(0, 2)$ and $(1, 3)$. All weak alternating languages of index $(0, 2)$ are G_δ sets. Taking into account, that the separation property for G_δ sets ([31, Theorem 22.16]) is essentially a triviality, it seems that the difficulty in establishing the separation property of the alternating hierarchy is probably not related to the topology or set theory, but rather to yet not discovered combinatorial or algebraic properties of automata on trees.

Papers not included in the series *Investigations of automata and related logics using methods of set theory.*

Algebraic characterization of Δ^0_2 sets. In the paper [20] we show that a minor modification of a technic developed in the paper [9] allows for characterization of Δ^0_2 sets in algebraic terms. The result of Bojańczyk and Place ([9]) concerns the Boolean combinations of open sets.

A paper related to the cardinal arithmetic and measure theory ([33]). Cardinal arithmetic has been an area of a systematic research from the inception of set theory and currently most questions are already answered. However, there is one important exception to this rule. The behavior of the mapping $\kappa \to 2^\kappa$ for cardinal κ of countable cofinality is still an open problem. This seemingly esoteric looking topic was developed by S. Shelah to the pcf [10] theory ([60, 59]). The pcf theory provides arguments, why cardinals such as $\aleph_\omega$ i $\aleph_{\omega+1}$, looking

[10] pcf abbreviates *possible cofinalities.*

from an appropriate perspective, can be considered much smaller than $2^{\aleph_0}$. One of the applications of the pcf theory is a proof, that there exists a *scale* of length $\aleph_{\omega+1}$ in the product

$$\prod_{n \in B} \aleph_n,$$

where $B \subseteq \omega$ is a specially chosen infinite subset of ω and the scale is a cofinal subset in the sense of the coordinatewise ordering modulo finite sets. The scale preserves many properties of the full product and at the same time is *much smaller*, because the whole product is of the size *at least* $2^{\aleph_0}$ and the scale has the size $\aleph_{\omega+1}$. In particular, the existence of a scale allows to *reduce* to the size $\aleph_{\omega+1}$ an important topological example of a normal space, such that $X \times [0, 1]$ is not normal (the example was constructed by M. E. Rudin [56], see also [34]). This example plays a certain role in abstract measure theory. Namely, J. Mařík ([40]) proved, that if $X \times [0, 1]$ is normal, then every Baire measure on X can be extended to a Borel measure on X. The σ–field of Borel sets is generated by all closed subsets of X and the σ–field of Baire sets is generated only by functionally closed sets, that is by preimages of points under continuous functions $f : X \to [0, 1]$.

In [33] we prove, that every topological space similar to one constructed in [34], has good measure-theoretic properties, that is every Baire measures can be extended to a Borel measure. The original, not downsized example of Rudin, also has good measure-theoretic properties, but we prove it only under a quite strong set-theoretical assumption. This line of research was motivated by a question of D. H. Fremlin on the limits of applicability of Mařík's theorem.

Four papers related to the descriptive set theory.

1. Hurewicz's theorem ([31, Theorem 21.18]) says that if an analytic set A in a Cantor set C is not an F_σ set, then one can find a standard witness for this fact, namely a copy of the Cantor set $D \subset C$ such, that $D \cap A$ is homeomorphic with the irrationals in the real line and $D \setminus A$ is homeomorphic with the rationals in the real line. Among further improvements of the theorem of Hurewicz, one of the most interesting is a result of Kechris, Louveau and Woodin ([31, Theorem 21.22]). In the paper [44] we show a topological version of the later result.

2. In the paper [43] I proved, using a method of J. Steel developed by F. van Engelen, that the space $\mathcal{K}(\mathbb{Q})$ of all compact subsets of the rationals with the Hausdorff metric is a homogeneous space, that is for every two points there exists a homeomorphism which swaps them.

3. In the paper [42] I proved, that the space $C_p(\mathcal{N})$, that is the space of all real-valued continuous functions on the irrationals with the topology of pointwise convergence, can be mapped in a continuous and bijective way on the Hilbert cube $[0, 1]^\omega$. In the construction the key observation is the fact, that $C_p(\mathcal{N})$ can be canonically mapped onto a coanalytic-complete set. The result answers questions of A. V. Archangielski and J. P. Christensen.

4. M. Morayne and C. Ryll-Nardzewski in the paper [47] asked a question about the characterization of functions $f : \mathbb{R} \to \mathbb{R}$ equivalent to Borel-measurable functions. In the paper [36] we give such a characterization based on an interesting result of J. Saint Raymond from [57]. Namely, Saint Raymond proved that the pair $(\mathcal{WF}, \mathcal{UB})$ is complete with respect to all disjoint pairs of coanalytic sets. The problem of completeness of a given pair of sets also appears in the paper (D).

Two papers from the general topology (three other papers from this area are skipped to save the space).

1. The paper [37] is about the σ–compactness and similar notions in the realm of topological groups. In the paper [37] we give an example which differentiates between two such notions. This answers questions of M. Tkachenko and C. Hernandez.

2. The paper [38] is about the Valdivia compact spaces. This is a generalization of Eberlein and Corson compacta, which are related to the weak topologies in the Banach spaces. In [38] we investigate the Valdivia compacta in the context of inverse limits of compact metric spaces and show as a corollary a preservation result about retractions and open mappings.

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