Some problems

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1.1 Exercises from Fulton-Harris §7

1.2 Let $G$ be a topological group. Show that if a subgroup of $G$ is open, then it is closed. Show that component of 1 is a subgroup.

1.3 Show that $\pi_1(G)$ is abelian.

1.4 Let $G$ be a connected topological group. Let $p: \tilde{G} \to G$ be a connected covering. Choose an inverse image of 1. Show that $\tilde{G}$ has a natural group structure, such that $p$ is a homomorphism. Interpret the kernel as the group of deck transformation.

1.5 Check basic properties of quaternions (octonions) e.g.: $\bar{a}a = |a|^2 \in \mathbb{R}$, $|ab| = |a||b|$ for $a, b \in \mathbb{H}$ (or $\in \mathbb{O}$).

1.6 Check that the set of the maps of algebras $\text{map}(C, H)$ is the 2-dimensional sphere.

1.7 Prove that $\text{Aut}(H) = \mathbb{H}^*/\mathbb{R}^*$.

1.8 Show that the only $\mathbb{R}$-division (associative) algebra are $\mathbb{C}$ and $\mathbb{H}$.

1.9 Show that $Sp(n) \subset SU(2n)$.

1.10 Show, that $Sp(n)$ is the maximal compact subgroup of $S\tilde{p}(n, \mathbb{C})$.

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2.1 Exercises from Fulton-Harris §8

2.2 Show that $\mathbb{R}^3$ with the vector product $\times$ is a Lie algebra isomorphic to $so(3)$.

2.3 Compare the Lie algebra of upper-triangular $3 \times 3$ matrices with 0’s on the diagonal with the Lie algebra generated by $x$ and $\frac{d}{dx}$ acting on the polynomial ring $\mathbb{C}[x]$.

2.4 Compute explicitly $\exp$ for the algebras above.

2.5 Check that the commutator of two derivations of an algebra (not necessarily associative) is a derivation.

2.6 For any $\mathbb{R}$-algebra compare $\text{Lie}(\text{Aut}(A))$ and $\text{Der}(A)$.
3

3.1 Compute $\text{Hom}(S^3, S^3)$.

3.2 Check that the complexifications of $U(n)$, $SU(n)$, $O(n)$, $SO(n)$, $Sp(n)$ are the groups $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$.

3.3 Compute Lie algebras of the classical compact groups, and find their complexifications.

3.4 Compute the differential of the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^2$ in the direction $X$. Show that it does not vanish if $A$ and $X$ are symmetric, $A$ positive definite.

3.5 Compute few terms of Baker-Campbell-Hausdorff formula. (At least the third term.)

3.6 Check the formula

$$\frac{d}{dt}e^{A+tB} = e^A \left( B - \frac{[A, B]}{2!} + \frac{[A, [A, B]]}{3!} - \frac{[A, [A, [A, B]]]}{4!} + \ldots \right).$$

3.7 Show that exp for $SU(2)$ is surjective. At which points it is a submersion?

3.8 Let $G \subset GL_n(\mathbb{C})$ be a reductive group. Define a hermitian product in $g$ by the formula $\langle (X, Y) \rangle = tr(XY^*)$. The hermitian product in $g$ allows to define the Cartan involution in $GL(g)$. Show that $Ad(G) \subset GL(g)$ is a reductive subgroup. (Show that $(ad_X)^* = ad_{X^*}$.)

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4.1 For which groups: $GL_n^+(\mathbb{R})$, $SL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $B_n$ (upper triangular), $N_n$ (upper triangular with 1’s on the diagonal) exp is surjective?

4.2 Compute the Killing form for the classical algebras (in particular for $sl_n(\mathbb{C})$, $gl_n(\mathbb{C})$) and for $b_n$. Show, that for $sl_n(C)$ the Killing form is equal up to a constant to $B_0(X, Y) = Tr(XY)$.

4.3 Let $g \subset End(\mathbb{C}[z])$ be the Lie subgroup generated by the multiplication by $z$ and $\frac{d}{dz}$. Show that $g$ has a finite dimension. Find a group $G$ which has Lie algebra $g$. Does $G$ act on $\mathbb{C}[z]$?

4.4 Exercises from Fulton-Harris §9.

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5.1 Compute what are the maximal tori in $U(n)$, $SU(n)$, $SO(n)$ and $Sp(n)$. What are the normalizers $N(T)$ and the Weyl groups.

5.2 Show that in $U(n)$ every commutative subgroup is included in a maximal torus.

5.3 Show that the same statement is not true for $SO(3)$.

5.4 Let $T$ be a torus (compact connected commutative Lie group). Show that there exists $g \in T$ such that $\langle g \rangle$ is dense in $T$.

5.5 Show that for any element $g$ of a topological group $G$ $\text{closure}(g)$ is abelian. For $G = U(n)$ characterize those elements for which $\text{closure}(g)$ is a maximal torus.

5.6 Jordan decomposition. Let a semisimple Lie algebra acts on a vector space $\rho : g \to End(V)$, and let $a \in g$. Decompose $\rho(a) = d + n$ where $d$ is diagonal in a certain basis, and $n$ is upper-triangular with 1’s at the diagonal. Show that there exist elements $ad$ and $an$ in $g$ such that $\rho(ad) = d$, $\rho(an) = n$ and $ad + an = a$. 

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6.1 Decompose $\text{Hom}(V, V)$ into irreducible representations of $G = \text{GL}(V)$, where $G$ acts on $\text{Hom}(V, V)$ – left multiplication
– conjugation.

6.2 Show that the natural representation of $\text{SL}_2(\mathbb{C})$ is isomorphic to its dual. (This is not true for $\text{GL}_2(\mathbb{C})$.)

6.3 Decompose bilinear forms on $V$ into irreducible representations of $\text{GL}(V)$.

6.4 Show that irreducible representations of $G \times H$ are of the form $V \otimes W$, where $V$ is a irreducible representation of $G$ and $W$ is a irreducible representation of $H$.

6.5 Let $V$ be irreducible real representation of odd dimension. Show that $V_{\mathbb{C}}$ is irreducible. If the dimension is even it can happen that $V_{\mathbb{C}} \simeq W \oplus \overline{W}$.

6.6 Show that two real representation are isomorphic if and only if their complexification are isomorphic.

6.7 Give the precise formula for the action of the Lie algebra $g$ on $\text{Hom}_G(V, W)$.

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7.1 Decompose into a direct sum of irreducible subrepresentations $\text{Sym}^3(\mathbb{C}^2) \otimes \text{Sym}^2(\mathbb{C}^2)$.

7.2 Show that $\text{Sym}^n(\text{Sym}^2(\mathbb{C}^2)) \simeq \bigoplus_{s=0}^{[n/2]} \text{Sym}^{2n-4s}(\mathbb{C}^2)$.

7.3 Decompose $\text{Sym}^2 \text{Sym}^3(\mathbb{C}^2)$.

7.4 What are the irreducible representations of $\text{GL}_2(\mathbb{C})$?

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Fulton-Harris: exercises §13–§14.

8.1 Decompose $\text{Sym}^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ into irreducible representations of $\text{SL}_2(\mathbb{C})$.

8.2 Show $\Lambda^{n-1} \mathbb{C}^n = (\mathbb{C}^n)^*$ as representations of $\text{SL}_n(\mathbb{C})$.

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Except from Fulton-Harris:

9.1 Show that for any simple Lie group $G$ the quotient $\Lambda/\Lambda_R$ is isomorphic to the center of $G$. Here $\Lambda$ is the lattice of weights, and $\Lambda_R$ is the sublattice generated by roots. First check the claim for $\text{SL}_n(\mathbb{C})$ and $\text{SO}(n)$.

9.2 Show that for a compact simple Lie group there exists only one up to a constant invariant scalar product, which is the –Killing form.

9.3 Check by examples $(\mathbb{C}^3, (\mathbb{C}^3)^*, \text{Sym}^2(\mathbb{C}^3)$, etc.) what is the kernel of the map $M(\omega) \rightarrow V(\omega)$ from the Verma module the to irreducible representation associated to a weight $\omega$. Then find a formula in a book.
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10.1 Find the maximal weight of the dual representation of the irreducible representation of \( SL_n(\mathbb{C}) \) corresponding to the diagram \( \lambda \).

10.2 Find Kostka numbers of the irreducible representation of \( SL_n(\mathbb{C}) \) corresponding to the diagram \( \lambda = (n-1, n-2, n-3, \ldots, 1, 0) \).

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11.1 Suppose that \( H \subset G \) and \( \text{rank } H = \text{rank } G \). Show that every root of \( H \) is a root of \( G \). Give interesting examples (\( GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \subset GL_{m+n}(\mathbb{C}) \) is a trivial example). Compute Weyl groups.

11.2 For \( SL_n(\mathbb{C}) \): can one split the homomorphism \( NT \to NT/T = W \)?

11.3 Construct a 2-fold coverings \( SU(2) \times SU(2) \to SO(4) \), \( Sp(2) \to SO(5) \), \( SU(4) \to SO(6) \).

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Exercises and examples from Fu-Ha SS16-17

12.1 Check how Weyl character formula works for \( sp(n) \).

12.2 Assume that in the category \( \mathcal{C} \) the isomorphism classes of objects forms a set \( X \). Assume that the direct sum exist. Define an relation \( X^2 \):

\[
([V], [W]) \sim ([V'], [W']) \quad \text{if} \quad \exists Z \in \text{Ob}(\mathcal{C}) \quad V \oplus W' \oplus Z \cong V' \oplus W \oplus Z.
\]

Check that it is an equivalence relation and that the in the set of equivalence classes one there is a natural stucture of an abelian group.

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13.1 „Bott periodicity” for complex Clifford algebras: Check that \( C_{n+2} \otimes_{\mathbb{R}} \mathbb{C} \) is isomorphic to the algebra of \( 2 \times 2 \) matrices with coefficients in \( C_n \otimes_{\mathbb{R}} \mathbb{C} \).

13.2 Compute the group of invertible elements \( C^*_2 \) of the real Clifford algebra and the Clifford group \( \Gamma_2 \). Which two circles in \( \Gamma_2 \) form \( Pin(2) \)?

13.3 Find explicite isomorphisms or show that it does not exist between representations of \( Spin(n) \):

- spinors \( S \) and \( S^* \) for \( n \) odd
- spinors \( S^\pm \) and \( (S^\pm)^* \) for \( n \) even

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14.1 Which spinors are complexifications of real representations of \( Spin(n) \)? The answer depends on the deisibility of \( n \) by 8.

14.2 Check the isomorphism of \( Spin(2n) \) representations

\[
\text{Sym}^2(S^+) = (\lambda^n)^+ + \lambda^{n-4} + \lambda^{n-8} + \ldots
\]

\[
\Lambda^2(S^+) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \ldots
\]

\[
\text{Sym}^2(S^-) = (\lambda^n)^- + \lambda^{n-4} + \lambda^{n-8} + \ldots
\]

\[
\Lambda^2(S^-) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \ldots
\]

Here \( \lambda^k \) is the \( k \)-th exterior power of the natural representation of the orthogonal group.
14.3 Check the isomprphism of $\text{Spin}(2n + 1)$ representations

\[ \text{Sym}^2(S) = \lambda^n + (\lambda^{n-3} + \lambda^{n-4}) + \ldots + (\lambda^{n-4i-3} + \lambda^{n-4i-4}) + \ldots \]

\[ \Lambda^2(S) = (\lambda^{n-1} + \lambda^{n-2}) + (\lambda^{n-5} + \lambda^{n-6}) + \ldots + (\lambda^{n-4i-1} + \lambda^{n-4i-2}) + \ldots \]

15.1 Show that the stabilizer in $GL_7(\mathbb{R})$ of the 3-form $\Phi \in \bigwedge^3 \mathbb{R}^n$ defining multiplication of octonions is equal $G_2$. 