MOTIVIC CHERN CLASSES AND K-THEORETIC STABLE ENVELOPES

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Abstract. We consider a smooth algebraic variety with an action of an algebraic group acting with finitely many orbits. We study equivariant characteristic classes of the orbits, namely the equivariant *motivic Chern classes*, in the K-theory of the ambient space. We prove that the motivic Chern class satisfies the axiom system inspired by that of "K-theoretic stable envelopes", recently defined by Okounkov and studied in relation with quantum group actions on the K-theory algebra of moduli spaces. We also give explicit formulas for the equivariant motivic Chern classes of Schubert cells and matrix Schubert cells. Lastly, we calculate the equivariant motivic Chern class of the orbits of the A_2 quiver representation, which yields formulas for the motivic Chern classes of determinantal varieties and more general degeneracy loci.

1. Introduction

Characteristic classes of singular varieties are important tools in algebraic and enumerative geometry. These classes live in some extraordinary cohomology theory, and even within the same theory there are a few different flavors of them. For example, in ordinary cohomology theory one has the *Chern-Schwartz-MacPherson* (CSM) class (together with its variants, the *Segre-Schwartz-MacPherson* (SSM) class, and the *characteristic cycle*), the *Fulton-Johnson* class, and the *Chern-Mather* class. In K-theory one has Brasselet-Schürmann-Yokura's *motivic Chern class* (or its cohomology shadow, the *Hirzebruch class*). The corresponding χ_y-genus is a common generalization of the L-genus and the Todd-genus.

In the present paper we define and study the *equivariant motivic Chern class*. After their definition in Section 2 we give three contributions to their theory and computation, detailed in the next three subsections.

1.1. **K-theoretic stable envelopes, motivic Chern classes.** We consider a smooth algebraic variety with an action of an algebraic group acting with finitely many orbits. We study equivariant motivic Chern classes of the orbits in the K-theory of the ambient space. In Theorems 5.3 and 5.5 we present an axiomatic characterization of the equivariant motivic Chern class. The axiom system is inspired by works of Okounkov.

More precisely, in [O, Section 9.1] characteristic classes called *K-theoretic stable envelopes* are associated with certain varieties, namely "full attracting sets" in Nakajima quiver varieties (see also [MO1, MO2, OS, AO1, AO2] and [GRTV, RTV1, RTV2, RTV3, RV1, RV2, FRV, SZZ]). Stable envelopes are defined by three axioms: a normalization, a support, and a Newton polygon axiom—and they also depend on an extra parameter called slope (or alcove). Our axioms determining the equivariant motivic Chern class (of an invariant subvariety of a G-variety with finitely many orbits), see Theorems 5.3, 5.5, are of the same nature: a normalization, a support, and a Newton polygon axiom. The first two are direct analogues of the axioms in [O, 9.1.3], but our Newton polygon axiom is *weaker*; it is implied by Okounkov's Newton polygon axiom (for a specific choice of slope), but it's not equivalent with it. Therefore, if for a variety stable envelopes are defined then for a particular choice of slope the stable envelope must coincide with the motivic Chern class.

The Białynicki-Birula decomposition plays a crucial role in our proof of the Newton-polygon axiom of motivic Chern classes. The asymptotic behavior of motivic Chern classes is related with the geometry of attracting sets with respect to one parameter subgroups of the torus.

Our Newton polygon axiom (Theorem 5.3[(iii)]) is weaker than Okounkov's for a reason: for generic varieties the stronger Newton polygon axiom [O, 9.1.8] does *not* hold, only our weak one. Yet, putting together this weak axiom with the normalization and support axioms they characterize the motivic Chern class.

In [RV2] (see also [AMSS]) an analogous result is proved in cohomology instead of K-theory: it is shown that the Chern-Schwartz-MacPherson classes are the same as the *cohomological* stable envelopes. As a result, the problem of calculating CSM classes is reduced from a genuinely commutative algebra problem to an interpolation (and hence essentially a linear algebra) problem. This drastic simplification of the calculation of CSM classes is used e.g. in [FR2, P, R].

Our Theorems 5.3 and 5.5 reduce the problem of calculating equivariant motivic Chern classes to an interpolation problem. Hence they open the way for finding motivic Chern classes of quiver orbits, matroids, and singularities.

1.2. **Motivic Chern classes of matrix Schubert varieties.** The theory of *cohomological fundamental classes* is well developed. For quiver representations the fundamental class is called quiver polynomial, for singularities of maps it is called Thom polynomial. In any of these areas and in others, it turns out that cohomological fundamental classes *should be* expressed in Schur polynomials—because those expansions show stability and positivity properties, e.g. [PW, BR]. Moreover, Schur polynomials are cohomological fundamental classes themselves, namely those of the so-called matrix Schubert varieties [FR1].

The theory of *K-theoretic fundamental classes* has a similar shape. The atoms of the theory are Grothendieck polynomials: K-theoretic fundamental classes in quiver and Thom polynomial settings show stability and positivity properties as soon as they are expanded in (stable double) Grothendieck polynomials [Bu, RSz]. Moreover, these Grothendieck polynomials are the K-theoretic fundamental classes of matrix Schubert varieties.

The theory of *CSM/SSM classes* (i.e. the theory of cohomological stable envelopes) also has a similar shape. The atoms of the theory are the $\widetilde{s}_\lambda$ classes of [FR2]: SSM classes of geometrically relevant varieties, when expanded in the $\widetilde{s}_\lambda$ classes show stabilization and positivity properties. Moreover, $\widetilde{s}_\lambda$ classes are the SSM classes of matrix Schubert cells.

Hence, it is natural to predict that the atoms of the theory of motivic Chern classes all over geometry will be the motivic Chern classes of matrix Schubert cells. In Theorems 6.4 and 7.4 we present formulas for the equivariant motivic Chern classes of Schubert cells in partial flag varieties and those of matrix Schubert cells. The formulas will be appropriate versions of the so-called trigonometric weight functions of Tarasov-Varchenko, whose role in Schubert calculus was first discovered in [RTV1].

1.3. **Determinantal varieties.** In Section 8 we study the motivic Chern classes of the orbits of $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ acted upon by $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. This representation is called the A_2 quiver representation, and after projectivization the orbit closures are called determinantal varieties. We prove formulas for the motivic Chern classes of the orbits, both in the traditional "localization" form, and also expanded in the building blocks invented in Section 7. These results can be viewed as the K-theory generalizations of [PP, FR2, Z].

In the paper there will be three concrete calculations of equivariant motivic Chern classes: those of Schubert cells, of matrix Shubert cells, and of A_2 orbits. These three calculations illustrate the three different methods know to the authors: calculation by interpolation axioms, by resolution with normal crossings, and by the sieve of general resolutions.

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2. The motivic Chern class

In this section we recall the notion of motivic Chern class from [BSY]. The images of motivic Chern classes in cohomology are called Hirzebruch classes. We will set up the equivariant versions of these classes. The equivariant versions of Hirzebruch classes were studied in [We2, We3].

2.1. **Defining invariants of singular varieties.** Many invariants of smooth compact algebraic varieties can be defined through some operation on the tangent bundle. It is natural to ask whether they can be extended to singular varieties, which of course do not have tangent bundles. The same question applies to open manifolds which do not possess a fundamental class, but can be compactified. One natural approach is to resolve singularities or compactify and apply the preferred invariant to the resolution or compactification. In general the result depends on the resolution/compactification. One can try to modify the result by correction terms coming from the exceptional or boundary divisors. This way Batyrev defined stringy Hodge numbers [Ba] for varieties with at worst Kawamata log-terminal singularities. Borisov-Libgober [BoLi] have defined the elliptic cohomology class for the same class of singularities. The construction extends to pairs, also with a group action. Taking a resolution one usually assumes that the exceptional locus is a sum of smooth divisors intersecting transversely. To prove that the invariant does not depend on the resolution it is enough to check that it does not change when we modify the resolution by blowing up a smooth center well posed with respect to the existing configuration of divisors. This is a consequence of Weak Factorization Theorem [Wl]. The construction can be carried out in the presence of a group action since by Bierstone and Milman [BiMi] equivariant resolutions exist and the Weak Factorization can be realized in the invariant manner. For example the

Chern-Schwartz-MacPherson class can be defined using this technique, as discovered by Aluffi. Moreover this way Aluffi obtained new results, similar to the invariance of Batyrev's stringy Hodge numbers.

We will apply the described method *to define* the equivariant motivic Chern class of singular varieties with arbitrary singularities. In later sections we will present three approaches to their calculation: the direct application of the definition, see Theorem 7.4, an axiomatic characterisation, see Theorems 5.3, 5.5 and Theorem 6.4, and the sieve method illustrated in Section 8.4.

2.2. **The** mC**-classes.** Consider the two covariant functors $\mathrm{Var}(-)$ and $K_{alg}(-)[y]$

$$\text{(smooth varieties, proper maps)} \longrightarrow \text{(groups, group homomorphisms)}.$$

Recall that $\mathrm{Var}(M)$ is generated by classes of maps $X \to M$, where X can be singular, and f is not necessarily proper, modulo additivity relations see [BSY]. Recall that $K_{alg}(M)[y]$ is the K-theory ring of M with y a formal variable. By K-theory we mean the Grothendieck group generated by locally free sheaves. The version of K-theory built from the coherent sheaves is denoted by $G(X)$. If X is smooth, then the natural map $K_{alg}(X) \to G(X)$ is an isomorphism. We should have in mind that the push-forward of a sheaf is in a natural way represented in $G(X)$ by a coherent sheaf, which is not necessarily locally free. In $K_{alg}(X)$ the class of this sheaf is represented by its resolution. The topological K-theory $K_{top}(X)$ is the Grothendieck group made from topological vector bundles. The natural map $K_{alg}(X) \to K_{top}(X)$ in general is not an isomorphism except some rare cases, for example when the space admits a decomposition into algebraic cells.

The motivic Chern class mC is a natural transformation

$$\mathrm{Var}(M) \to K_{alg}(M)[y].$$

Thus for a map $X \to M$ we have $\mathrm{mC}([X \to M]) \in K_{alg}(M)[y]$. When there is no confusion we will drop the brackets from the notation and write only $\mathrm{mC}(X \to M)$, or the even simpler $\mathrm{mC}(X)$ if $X \subset M$. The transformation of functors mC is the unique *additive* transformation satisfying the normalization condition

$$\mathrm{mC}(\mathrm{id}_M) = \lambda_y(T^*M), \tag{1}$$

where for a vector bundle E we define $\lambda_y(E) := \sum_{i=0}^{\mathrm{rank}\, E} [\Lambda^i E] y^i$.

In [BSY] singular spaces M were also allowed at the cost of replacing $K_{alg}(M)$ with the K-theory of coherent sheaves $G(X)$. We will not need that generality in this paper, and hence we will not use the notation $G(X)$ any more.

2.3. **The equivariant** mC **class.** Let G be an algebraic linear group. We will use the G-equivariant version of mC: for a smooth G-space M and a G-equivariant map $f : X \to M$ of G-varieties we assign an element $\mathrm{mC}(f) = \mathrm{mC}(X \to M)$ in $K^G_{alg}(M)[y]$. This assignment is additive and satisfies the normalization property (1) now read equivariantly. The class is determined by the three properties:

Additivity: If $X = Y \sqcup U$, then $\mathrm{mC}(X \to V) = \mathrm{mC}(Y \to V) + \mathrm{mC}(U \to V)$.

Functoriality: For $f : V \to V'$ we have $\mathrm{mC}(X \stackrel{f\circ g}{\to} V') = f_* \mathrm{mC}(X \stackrel{g}{\to} V)$.

Normalization: We have $\mathrm{mC}(\mathrm{id}_V)=\lambda_y(T^*V)$.

By $K^G_{alg}(M)[y]$ we mean the equivariant K-theory built from G-equivariant locally free sheaves on M. Since M is smooth the map from the equivariant K-theory of coherent sheaves is an isomorphism. (See the argument of [Nie, Prop. 2.1] which applies for any algebraic linear group.) Therefore we have well defined push-forward to the equivariant K-theory of a smooth variety.

There are numerous ways to argue for the existence of the equivariant version of motivic Chern classes. The most straightforward way is repeating (with some care) the arguments of [BSY] line by line in the equivariant setting; via approximating classifying spaces with their finite dimensional skeletons, analogously to how Ohmoto defined the equivariant version of Chern-Schwartz-MacPherson classes in [O1, O2]. In fact this program was carried out in [We2, We3] for Hirzebruch classes. We do **not** follow this procedure here, rather we directly construct an element in equivariant K-theory represented by an equivariant sheaf. Moreover, we do not use here classifying spaces. Thus the obtained invariant is more refined, since the map

$$K^G_{top}(X)\to K_{top}(EG\times_G X)$$

is an isomorphism only after the completion in the *dimension ideal.* See the discussion for topological K-theory in Section 9.

2.4. **Definition of** $\mathrm{mC}(U\to X)$ **for smooth** U**.** The usual way of defining motivic Chern classes is using a special resolution.

Definition 2.1. *Suppose $f:U\to X$ is a map of smooth G-varieties. Then a* proper normal crossing extension *of f is a proper map $\bar f:Y\to X$ with an embedding $\iota:U\hookrightarrow Y$ satisfying $f=\bar f\circ\iota$, such that the variety Y is smooth and the compement $Y\setminus\iota(U)=\bigcup_{i=1}^s D_i$ is a smooth normal crossing divisor.*

As discussed in [We3, §5] such proper normal crossing extension always exists.

Now suppose that $f:U\to X$ is a map of smooth G-varieties and let $\bar f:Y\to X$ a proper normal crossing extension of f. For $I\subset\underline{s}=\{1,2,\dots,s\}$ let $D_I=\bigcap_{i\in I}D_i$, $f_I=\bar f|D_I$, in particular $f_\emptyset=\bar f$. Then

$$\mathrm{mC}(f)=\sum_{I\subset\underline{s}}(-1)^{|I|}f_{I*}\lambda_y(T^*D_I)\,. \tag{2}$$

The independence of the chosen extension follows from a refined version of Weak Factorization, as formulated in [Wl]. Here is a sketch of the arguments, essentially a repetition of the proof of [Bi, Th. 5.1]. Suppose that Y and Y' are two proper normal crossing extensions of U. By [Wl, Th. 0.0.1] we can assume that Y' is obtained by a blowup of Y in a smooth G-invariant center C which has normal crossing[1] with $D=Y\setminus U$. Let $\sigma:Y'\to Y$ be a blow-down. Let $E\subset Y'$

[1]A smooth subvariety $C\subset Y$ has normal crossing with a divisor D if at each point there exist a system of local coordinates such that the components of the divisor containing this point are given by the vanishing of one of the coordinates, and C is given by the vanishing of a set of coordinates.

be the exceptional divisor and D'_I be the proper transform of D_I. Comparing the formulas corresponding to the two extensions we see that it remains to check that

$$\sigma_*(\lambda_y(T^*D'_I)) - \sigma_*(\lambda_y(T^*(D'_I \cap E))) = \lambda_y(T^*D_I) - i_*\lambda_y(T^*(D_I \cap C),$$

where $i : D_I \cap C \to C$ is the inclusion. All the different restrictions of σ are denoted by σ to keep the notation short.

The equality means that for each integer p

$$\sigma_*(\Omega^p_{D'_I}) - \sigma_*(\Omega^p_{D'_I \cap E}) = \Omega^p_{D_I} - \Omega^p_{D_I \cap C}.$$

(We use the notation Ω^p for Λ^p to emphasize that we are pushing forward sheaves.) The right hand side has no higher cohomology. The equality in K-theory follows from the natural G-equivariant isomorphisms

$$\begin{aligned} \Omega^p_{D_I} &\to R^0\sigma_*(\Omega^p_{D'_I}), \\ \Omega^p_{D_I \cap C} &\to R^0\sigma_*(\Omega^p_{D'_I \cap E}) \quad \text{projective bundle}, \\ R^k\sigma_*(\Omega^p_{D'_I}) &\to R^k\sigma_*(\Omega^p_{D'_I \cap E}) \quad \text{for } k > 0, \end{aligned}$$

where $R^k\sigma_*$ is the k-th derived push forward. The isomorphisms can be checked locally, see [GNA, Prop. 3.3].

Remark 2.2. If X is not smooth then we would have to use the K-theory of coherent sheaves, not necessarily locally free, as it is done in [BSY, Cor. 0.1] in the non-equivariant case.

Since we use topological methods we pass to the topological equivariant K-theory defined by Segal [Seg] for compact groups. From now on by $K^G(X)$ we mean the Segal topological K-theory of the maximal compact subgroup of G. Obviously $K^G_{alg}(X)$ maps to $K^G_{top}(X)$. By $\mathrm{mC}(X \to M)$ we will understand the image of the equivariant motivic Chern class in the equivariant K-theory in the sense of Segal for the maximal compact subgroup of G. Basic information about Segal's K-theory is given in Section 9.

2.5. **The χ_y and equivariant χ_y genera.** The mC class of a map $X \to \mathrm{pt}$ to a point is an element of the representation ring

$$\chi^G_y(X) := \mathrm{mC}(X \to \mathrm{pt}) \in K^G(\mathrm{pt})[y] = R(G)[y],$$

and it is called the (equivariant) χ_y-genus of X. If $G = \mathbb{T}$ is a torus then $\chi^{\mathbb{T}}_y(X)$ contains no information about the action of G:

$$\chi^{\mathbb{T}}_y(X) = \iota(\chi_y(X)), \quad \text{where} \quad \iota : \mathbb{Z}[y] = R(\{1\})[y] \to R(\mathbb{T})[y].$$

This property is called the rigidity of the χ_y-genus, [We2, Th. 7.2], see also [Mus]. The same applies to connected reductive groups for which $R(G) = R(\mathbb{T})^W$, where $\mathbb{T}$ is the maximal torus of G and W is the Weyl group.

2.6. **Notation conventions.** Let $\mathbb{C}^*$ act on $\mathbb{C}$ by $\xi \cdot x = \xi^k x$, and let ℓ be the $\mathbb{C}^*$-equivariant line bundle over the point with this action. We will use the notation

$$K^{\mathbb{C}^*}(\mathbb{C}) = K^{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{Z}[\xi^{\pm 1}], \qquad [\ell] = k \cdot \xi, \qquad \text{and hence} \qquad \lambda_y(\ell) = 1 + y/\xi^k.$$

That is, by slight abuse of notation we use the same (Greek) letter for the coordinate of $\mathbb{C}^*$ and the generator of $K^{\mathbb{C}^*}(\mathrm{pt})$.

The map $i^* : K^G(X) \to K^G(U)$ in K-theory induced by the embedding $i : U \subset X$ will be called a restriction map and will be denoted by $f \mapsto f|U$.

2.7. **The fundamental calculation.** The results of the following basic calculation will be essential in later sections. Let a torus $\mathbb{T} = (\mathbb{C}^*)^r$ act on $\mathbb{C}$, and let the class of this equivariant line bundle over a point be $\alpha \in K^{\mathbb{T}}(\mathrm{pt})$. Using the notational conventions above, and the additivity of the motivic Chern class we have

$$\mathrm{mC}(\{0\} \subset \mathbb{C}) = 1 - 1/\alpha, \qquad \mathrm{mC}(\mathbb{C} \subset \mathbb{C}) = 1 + y/\alpha, \tag{3}$$
$$\mathrm{mC}(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + y/\alpha) - (1 - 1/\alpha) = (1 + y)/\alpha.$$

Remark 2.3. Those familiar with the Chern-Schwartz-MacPherson classes c^{sm} in equivariant cohomology may find it instructive to compare (3) with the analogous expressions

$$\mathrm{c}^{\mathrm{sm}}(\{0\} \subset \mathbb{C}) = a, \qquad \mathrm{c}^{\mathrm{sm}}(\mathbb{C} \subset \mathbb{C}) = 1 + a, \qquad \mathrm{c}^{\mathrm{sm}}(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + a) - (a) = 1.$$

The CSM class can be obtained from the formulas in (3) by applying the following operations

- substitute $\alpha = \exp(at)$, $y = -\exp(-ht)$;
- take the coefficient of t^d in the Taylor expansion in t, where d is the dimension of the ambient space;
- substitute $h = 1$;

while the class before substituting $h = 1$ is called the characteristic cycle class, or cohomological stable envelope, see [FR2, AMSS] and references therein. We note that the passage from Hirzebruch class to CSM class was given by Yokura, [Yo1, Rem. 2] by certain substitution and specialization to $y = -1$. Our procedure is a variant of that.

3. Newton polygons and N-smallness

In this section we consider the ring of Laurent polynomials in r variables $\mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}]$. This ring will enter our geometric calculations below as the $(\mathbb{C}^*)^r$-equivariant K-theory algebra of a point.

3.1. **Convex polygons and their projections.** A convex polygon in $\mathbb{R}^r$ is the convex hull of finitely many points in $\mathbb{R}^r$. For convex polygons $U, V \subset \mathbb{R}^r$ we have the Minkowski sum operation $U + V = \{u + v \mid u \in U, v \in V\}$. It is known that this operation (on convex polygons) satisfies the *Cancellation Law*

$$U \subset W \ \Leftrightarrow\ U + V \subset W + V. \tag{4}$$

For $s = (s_1, \dots, s_r) \in \mathbb{R}^r$ define the linear map $\pi_s : \mathbb{R}^r \to \mathbb{R}$ by $\pi_s(u_1, \dots, u_r) = s_1u_1 + s_2u_2 + \dots + s_ru_r$. The next proposition follows from the general theory of supporting hyperplanes of convex polygons.

Proposition 3.1. *Let U and V be convex polygons in $\mathbb{R}^r$. Let K be a union of finitely many hyperplanes in $\mathbb{R}^r$. The following conditions are equivalent:*

- $U \subset V$;
- $\pi_s(U) \subset \pi_s(V)$ *for all* $s \in \mathbb{R}^r$;
- $\pi_s(U) \subset \pi_s(V)$ *for all* $s \in \mathbb{Z}^r - K$.

Moreover, if $U \subsetneq V$, then there exists an $s \in \mathbb{Z}^r - K$ such that $\pi_s(U) \subsetneq \pi_s(V)$. □

3.2. **Newton polygons of Laurent polynomials and their toric substitutions.** For an element $f \in \mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}]$ we define its Newton polygon $\mathcal{N}(f)$ to be the convex hull of the finite set

$$\{(u_1, \dots, u_r) \in \mathbb{Z}^r \mid \text{the coefficient of } \alpha_1^{u_1} \dots \alpha_r^{u_r} \text{ in } f \text{ is not } 0\}$$

in $\mathbb{R}^r$. We have

$$\mathcal{N}(fg) = \mathcal{N}(f) + \mathcal{N}(g), \qquad \mathcal{N}(f+g) \subset conv.hull(\mathcal{N}(f), \mathcal{N}(g)). \tag{5}$$

Definition 3.2. *For $s = (s_1, \dots, s_r) \in \mathbb{Z}^r$ consider the map (called one parameter subgroup) $\kappa_s : \mathbb{C}^* \to (\mathbb{C}^*)^r$, $\xi \mapsto (\xi^{s_1}, \dots, \xi^{s_r})$. The induced ring homomorphism $\mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}] \to \mathbb{Z}[\xi^{\pm 1}]$ will be called the $\alpha = \xi^s$ toric substitution, and will be denoted by $f(\alpha) \mapsto f(\xi^s)$ or $f(\alpha) \mapsto f(\alpha)|_{\alpha = \xi^s}$.*

Now we discuss the relation between the projection π_s of Section 3.1 and the toric substitution.

Proposition 3.3. *Let $f, g \in \mathbb{Z}[\alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}]$.*

- *There exists a finite union of hyperplanes $K \subset \mathbb{Z}^r$ such that*
$$\mathcal{N}(f) \subset \mathcal{N}(g) \qquad \Longrightarrow \qquad \mathcal{N}(f(\xi^s)) \subset \mathcal{N}(g(\xi^s)) \ \forall s \in \mathbb{Z}^r - K.$$
- *If $K \subset \mathbb{Z}^r$ is a finite union of hyperplanes then*
$$\mathcal{N}(f(\xi^s)) \subset \mathcal{N}(g(\xi^s)) \ \forall s \in \mathbb{Z}^r - K \qquad \Longrightarrow \qquad \mathcal{N}(f) \subset \mathcal{N}(g).$$
- *Let $K \subset \mathbb{Z}^r$ be a finite union of hyperplanes. If $\mathcal{N}(f) \subsetneq \mathcal{N}(g)$ then there is an $s \in \mathbb{Z}^r - K$ such that $\mathcal{N}(f(\xi^s)) \subsetneq \mathcal{N}(g(\xi^s))$.*

Proof. The statements follow from Proposition 3.1 after observing that

$$\mathcal{N}(f(\xi^s)) \begin{cases} \subset \pi_s(\mathcal{N}(f)) & \text{for all } s \in \mathbb{Z}^r, \\ = \pi_s(\mathcal{N}(f)) & \text{for } s \in \mathbb{Z}^r - K, \end{cases} \tag{6}$$

where K is a finite union of hyperplanes. □

3.3. The N-smallness property.

Definition 3.4. *Let a rational function $h(\alpha_1,\dots,\alpha_r)$ be written as the ratio f_1/f_2, with $f_1,f_2\in\mathbb{Z}[\alpha_1^{\pm1},\dots,\alpha_r^{\pm1}]$. We say that h is N-small (N for "Newton polygon"), if $\mathcal{N}(f_1)\subset\mathcal{N}(f_2)$.*

Observe that the definition makes sense. Indeed, if h is presented in two different ways as a ratio, then they are necessary $f_1/f_2=(f_1\cdot g)/(f_2\cdot g)$; and claims (4) and (5) imply that one presentation is N-small if and only if the other presentation is N-small.

Lemma 3.5. *The set of N-small rational functions is closed under addition and multiplication.*

Proof. Let f_1/f_2 and g_1/g_2 be N-small. We have $\mathcal{N}(f_1g_2)=\mathcal{N}(f_1)+\mathcal{N}(g_2)\subset\mathcal{N}(f_2)+\mathcal{N}(g_2)=\mathcal{N}(f_2g_2)$. Similarly $\mathcal{N}(f_2g_1)\subset\mathcal{N}(f_2g_2)$. Then (5) implies $\mathcal{N}(f_1g_2+f_2g_1)\subset\mathcal{N}(f_2g_2)$ which proves the claim for addition. The claim for multiplication is proved by $\mathcal{N}(f_1g_1)=\mathcal{N}(f_1)+\mathcal{N}(g_1)\subset\mathcal{N}(f_2)+\mathcal{N}(g_2)=\mathcal{N}(f_2g_2)$. □

The following analytic characterization of N-smallness of rational functions will be useful.

Proposition 3.6. *Let $h(\alpha_1,\dots,\alpha_r)$ be a rational function.*

- *There exists a finite union of hyperplanes $K\subset\mathbb{Z}^r$ such that*

$$h \text{ is N-small} \qquad\Longrightarrow\qquad \lim_{\xi\to\infty}h(\xi^s) \text{ is finite for all } s\in\mathbb{Z}^r-K.$$

- *Let $K\subset\mathbb{Z}^r$ be a finite union of hyperplanes. Then*

$$\lim_{\xi\to\infty}h(\xi^s) \text{ is finite for all } s\in\mathbb{Z}^r-K \qquad\Longrightarrow\qquad h \text{ is N-small.}$$

Proof. The statements follow from Proposition 3.3 and elementary calculus knowledge about the limits of rational functions in one variable. □

4. Newton polygon properties of motivic Chern classes of subvarieties

We recall the notion of the Białynicki-Birula cell which was originally defined for $\mathbb{C}^*$, but we need it for a one parameter subgroup of $\mathbb{C}^*\to\mathbb{T}$.

Let $\mathbb{T}=(\mathbb{C}^*)^r$ act on a variety V, and consider the one parameter subgroup $\kappa_s:\mathbb{C}^*\to\mathbb{T}$ of Definition 3.2. Let F be a connected component of the fixed point set $V^{\kappa_s(\mathbb{C}^*)}$. The Białynicki-Birula cell is the subvariety

$$V_F^s=\left\{x\in V\ :\ \lim_{\xi\to\infty}(\kappa_s(\xi)\cdot x)\in F\right\}.$$

If $\mathbb{T}=\mathbb{C}^*$, $s=1$, then $\kappa_1=id$ and V_F^s coincides with the minus-cell V_F^- defined in [B1]. If $0\in V$ is an isolated fixed point we will write V_0^s instead of $V_{\{0\}}^s$.

Remark 4.1. If V is a vector space with a linear action of $\mathbb{T}$, then $V_0^s\subset V$ is the subspace spanned by the weight vectors with negative weights with respect to the $\mathbb{C}^*$-action induced by κ_s.

The Białynicki-Birula decomposition was studied recently by Drinfeld and Gaitsgory [DG] with an application to derived categories. In their construction the functorial properties of the stable and unstable sets for a $\mathbb{G}_m$–action play a crucial role. We need the functorial properties of the Białynicki-Birula decomposition to control the asymptotic behaviour of the motivic Chern classes.

Theorem 4.2. *Let V be a smooth variety with $\mathbb{T} = (\mathbb{C}^*)^r$ action. Suppose $0 \in V$ is an isolated fixed point. Let $\Sigma \subset V$ be an invariant subvariety, not necessarily closed. Then*

$$\mathcal{N}(\mathrm{mC}(\Sigma \subset V)|0) \subset \mathcal{N}(\lambda_{-1}(T_0^*V)) .$$

Moreover for almost all $s \in \mathbb{Z}^r$ (i.e. the set of exceptions is contained in a finite union of hyperplanes) we have

$$\lim_{\xi\to\infty}\left(\frac{\mathrm{mC}(\Sigma \subset V)|0}{\lambda_{-1}(T_0^*V)}\bigg|_{\alpha=\xi^s}\right) = \chi_y(\Sigma \cap V_0^s) .$$

Remark 4.3. For a non-isolated fixed point component F the analogous statement is

$$\lim_{\xi\to\infty}\left(\frac{\mathrm{mC}(\Sigma \subset V)|F}{\lambda_{-1}(\nu_F^*)}\bigg|_{\alpha=\xi^s}\right) = \mathrm{mC}(\Sigma \cap V_F^s \to F) \in K^{\mathbb{C}^*}(F)[y] .$$

The proof is a direct application of Proposition 3.1 and [We3, Theorem 7]. For the convenience of the reader here we reformulate the proof of [We3] with some simplifications and the notation adapted to our situation.

Proof. By the additivity of mC we can assume that Σ is smooth. According to Proposition 3.6 it is enough to prove that

$$\lim_{\xi\to\infty}\left(\frac{\mathrm{mC}(\Sigma \subset V)|0}{\lambda_{-1}(T_0^*V)}\bigg|_{\alpha=\xi^s}\right) < \infty. \tag{7}$$

for almost all $s \in \mathbb{Z}^r$, since $\mathcal{N}(\mathrm{mC}(\Sigma \subset V)|0_{|\alpha=\xi^s}) = \pi_s(\mathcal{N}(\mathrm{mC}(\Sigma \subset V)|0))$ for a generically chosen one parameter subgroup, c.f. Proposition 3.3.

Let $s \in \mathbb{Z}^r$ be such that the $\alpha = \xi^s$ substitution does not map any of the weights of the representation to 0; the thus excluded s vectors are contained in a finite union of hyperplanes.

The quantity $(\mathrm{mC}(\Sigma \subset V)|0/\lambda_{-1}(T_0^*V))|_{\alpha=\xi^s}$ can be interpreted as $\mathrm{mC}(\Sigma \subset V)|0/\lambda_{-1}(T_0^*V)$ for the $\mathbb{C}^*$-action obtained from the $\mathbb{T}$-action of the theorem by composing it with κ_s. Hence, in the remaining of the proof we work in $\mathbb{C}^*$-equivariant K-theory (without indicating this in our notation), in particular $K^{\mathbb{C}^*}(\{0\}) = \mathbb{Z}[\xi^{\pm1}]$ and want to show that

$$\lim_{\xi\to\infty}\frac{\mathrm{mC}(\Sigma \subset V)|0}{\lambda_{-1}(T_0^*V)} < \infty.$$

According to our assumption on s, the $\mathbb{C}^*$ action has a unique fixed point, 0.

Let Y be a proper normal crossing extension of the inclusion $\Sigma \to V$ (see Definition 2.1), i.e. Y is a smooth variety with $\mathbb{C}^*$ action, with an equivariant proper map $\eta : Y \to V$, such that $\eta^{-1}(\Sigma) \to \Sigma$ is an isomorphism and $Y \setminus \eta^{-1}(\Sigma)$ is a normal crossing divisor. For convenience let us assume that the map $Y \to V$ extends to a proper normal crossing extension $\hat\eta : \hat Y \to \hat V$ to a smooth compactification $\hat V$ of V. The complement of $\hat\eta^{-1}(\Sigma)$ is a normal crossing divisor denoted by $D = \bigcup_{i=1}^k D_i$. Then

$$\mathrm{mC}(\hat Y \subset \hat V) = \sum_{I\subset\{1,2,\dots,k\}} (-1)^{|I|}\eta_{I*}(\lambda_y(T^*D_I))\,, \tag{8}$$

where $D_I = \bigcap_{i\in I} D_i$, $D_\emptyset = \hat Y$ and η_I is the restriction of $\hat\eta$ to D_I. First we show that

$$\frac{(\eta_{I*}(\lambda_y(T^*D_I)))\,|0}{\lambda_{-1}(T^*_0V)}$$

is N-small. Since $V^{\mathbb{C}^*} = \{0\}$ the normal bundle to $V^{\mathbb{C}^*}$ is equal to T_0V. By the localization theorem in K-theory (compare [BFQ])

$$\frac{(\eta_{I*}(\lambda_y(T^*D_I)))|0}{\lambda_{-1}(T^*_0V)} = \sum \eta_{F*}\left(\frac{\lambda_y(T^*D_I)|F}{\lambda_{-1}(\nu^*_{F/D_I})}\right)\,. \tag{9}$$

The summation at the right hand side is over the components $F \subset D_I^{\mathbb{C}^*}$ such that $\eta(F) = \{0\}$. Here ν_{F/D_I} is the normal bundle to the fixed point set component in D_I. To compute the limit we first evaluate it in $K^{\mathbb{C}^*}(F)[y]$. The variety D_I is smooth and we obtain

$$\lim_{\xi\to\infty} \frac{\lambda_y(T^*D_I)|F}{\lambda_{-1}(\nu^*_{F/D_I})} = (-y)^{n_F}\lambda_y(T^*F), \tag{10}$$

where n_F is equal to the dimension of the subbundle of ν_{F/D_I} with negative weights. This holds because

$$\lim_{\xi\to\infty}\frac{1+y\xi^s}{1-\xi^s} = \begin{cases} -y & \text{for } s>0\\ 1 & \text{for } s<0,\end{cases}$$

see more details in [We3, Th. 13]. We conclude that the limit of the push-forward, which is the push-forward of the limit, is finite. This proves our first claim.

Now we compute the limit explicitly. Consider the Białynicki-Birula cell $(D_I)_F := (D_I)^s_F$ of F in D_I for our $\mathbb{C}^*$-action κ_s. Observe that

$$\chi_y((D_I)_F) = (-y)^{n_F}\chi_y(F) \tag{11}$$

since by [B1] the limit map $(D_I)_F \to F$ is a locally trivial fibration in Zariski topology and the fiber is $\mathbb{C}^{n_F}$. Hence the limit (10) is equal to $\chi_y((D_I)_F)$. We have

$$\bigcup_{F\subset D_I^{\mathbb{C}^*},\ \hat\eta(F)=\{0\}} (D_I)_F \;=\; \hat\eta_I^{-1}(V^s)\,.$$

Since $\hat\eta$ restricted to the inverse image of Σ is an isomorphism, therefore

$$\chi_y(V^s_0\cap\Sigma) = \sum_{I\subset\underline{k}}(-1)^{|I|}\chi_y(\eta_I^{-1}(V^s_0)) = \sum_{I\subset\underline{k}}(-1)^{|I|}\sum_{F\subset D_I^{\mathbb{C}^*},\ \hat\eta(F)=\{0\}}\chi_y((D_I)_F).$$

By (8–11) the former sum is the limit of $\mathrm{mC}(\Sigma\subset V)|0/\lambda_{-1}(T_0^*V)$ in $\mathbb{C}^*$-equivariant K-theory. This completes the proof. □

We will be concerned with a property of torus actions, which we will call *positive*, c.f. [FP].

Definition 4.4. *We call a representation of* $\mathbb{T}=(\mathbb{C}^*)^r$ positive*, if any of the following equivalent conditions are satisfied.*

- *There exists a one parameter subgroup $\alpha=\xi^s$ of $\mathbb{T}$ acting with positive weights;*
- *the weights of the $\mathbb{T}$-action are contained in an open half-space $\{u\in\mathbb{R}^r:\pi_s(u)>0\}$;*
- *the convex hull of the weights of the $\mathbb{T}$-action does not contain $0\in\mathbb{R}^r$.*

We call a G-representation positive *if its restriction to a maximal torus is positive.*

Corollary 4.5. *If the action of $\mathbb{T}=(\mathbb{C}^*)^r$ on V is positive and $0\notin\Sigma$, then*

$$\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0)\subset\mathcal{N}(\lambda_{-1}(T_0^*V))\setminus\{0\}.$$

Proof. The containment in $\mathcal{N}(\lambda_{-1}(T_0^*V))$ is explicitly stated in Theorem 4.2 (even without the conditions on the action and on Σ), we need to prove that $0\notin\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0)$. We will prove this by showing that 0 is not in a projection of $\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0)$.

We have $\lambda_{-1}(T_0^*V)=\prod(1-1/w_i)$ where w_i are the (multiplicatively written) weights of the action. Let $s\in\mathbb{Z}^r$ be such that the $\alpha=\xi^s$ substitution proves that the action is positive, and

$$\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0|_{\alpha=\xi^s})=\pi_s(\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0)). \tag{12}$$

Such a choice is possible because the s vectors proving positivity is open in the appropriate sense, and the s vectors for which (12) fails is a finite union of hyperplanes (c.f. Proposition 3.3).

We have $\lambda_{-1}(T_0^*V)|_{\alpha=\xi^s}=\prod(1-1/\xi^{k_i})$, where k_i are positive integers. Therefore

$$\mathcal{N}((\lambda_{-1}(T_0^*V))|_{\alpha=\xi^s})=\Big[-\sum k_i,0\Big]\subset\mathbb{R}_{\le 0}\,. \tag{13}$$

Note that, since the action through κ_s has positive weights, the Białynicki-Birula cell of 0 consists only of 0:

$$V_0^s=\{0\}\,.$$

By Theorem 4.2

$$\lim_{\xi\to\infty}\left(\left.\frac{\mathrm{mC}(\Sigma\subset V)|0}{\lambda_{-1}(T_0^*V)}\right|_{\alpha=\xi^s}\right)=\ \chi_y(\Sigma\cap V_0^s)=0\,,$$

since $\Sigma\cap V_0^s=\emptyset$. Using (13) this implies that $0\notin\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0|_{\alpha=\xi^s})$, and hence by (12) we have $0\notin\pi_s(\mathcal{N}(\mathrm{mC}(\Sigma\subset V)|0))$, what we wanted to prove. □

The next two examples illustrate different aspects of Corollary 4.5.

Example 4.6. Consider the *positive* action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$ given by $(\alpha,\beta)\cdot(x,y)=(\alpha\beta x,\alpha^3\beta^{-2}y)$; for example $s=(1,0)$ verifies that the action is positive. Let X and Y denote the first and the second coordinate axes respectively. The additivity and normalization properties of mC imply

$$\mathrm{mC}(\mathbb{C}^2)=(1+\frac{y}{\alpha\beta})(1+\frac{y}{\alpha^3\beta^{-2}}),\quad \mathrm{mC}(X)=(1+\frac{y}{\alpha\beta})(1-\frac{1}{\alpha^3\beta^{-2}}),\quad \mathrm{mC}(Y)=(1-\frac{1}{\alpha\beta})(1+\frac{y}{\alpha^3\beta^{-2}}),$$

$$\mathrm{mC}(\{0\}) = (1-\frac{1}{\alpha\beta})(1-\frac{1}{\alpha^3\beta^{-2}}), \qquad \mathrm{mC}(X-\{0\}) = \frac{1+y}{\alpha\beta}(1-\frac{1}{\alpha^3\beta^{-2}}),$$

$$\mathrm{mC}(Y-\{0\}) = (1-\frac{1}{\alpha\beta})\frac{1+y}{\alpha^3\beta^{-2}}, \qquad \mathrm{mC}(\mathbb{C}^2-\{0\}) = (y^2-1)\frac{1}{\alpha^4\beta^{-1}}+(y+1)\frac{1}{\alpha\beta}+(y+1)\frac{1}{\alpha^3\beta^{-2}}.$$

It is instructive to verify that the Newton polygons of $\mathrm{mC}(X-\{0\})$, $\mathrm{mC}(Y-\{0\})$, $\mathrm{mC}(\mathbb{C}^2-\{0\})$, namely the convex hulls of $\{(-4,1),(-1,-1)\}$, $\{(-4,1),(-3,2)\}$, $\{(-4,1),(-1,-1),(-3,2)\}$ are contained in $\mathcal{N}(\lambda_{-1}(\mathbb{C}^{2*}))\setminus\{0\} = \mathcal{N}(\mathrm{mC}(\{0\}))\setminus\{0\} = \mathrm{conv}(\{(-4,1),(-1,-1),(-3,2),(0,0)\}\setminus\{0\}$, as illustrated in Figure 1.

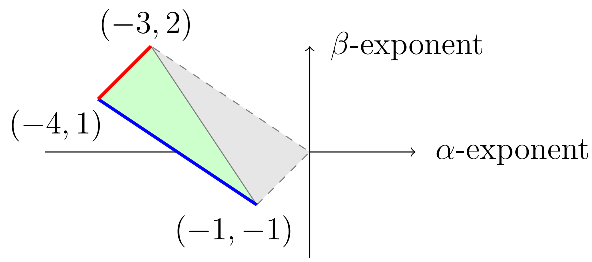


Figure 1. Newton polygons of Example 4.6

Example 4.7. If the action of $\mathbb{T}$ is not positive, then the inclusion of Corollary 4.5 may not hold. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $\alpha\cdot(x,y) = (\alpha x, \alpha^{-1}y)$. We have

$$\mathrm{mC}(\mathbb{C}^2-\{0\}) = (1+\frac{y}{\alpha})(1+\frac{y}{\alpha^{-1}}) - (1-\frac{1}{\alpha})(1-\frac{1}{\alpha^{-1}}) = (y+1)\frac{1}{\alpha}+(y^2-1)+(y+1)\alpha.$$

Hence, its Newton polygon is $[-1,1]$, which is not contained in $\mathcal{N}(\lambda_{-1}(\mathbb{C}^{2*}))\setminus\{0\} = \mathcal{N}((1-\frac{1}{\alpha})(1-\frac{1}{\alpha^{-1}}))\setminus\{0\} = [-1,1]\setminus\{0\}$. Note that here $\mathbb{C}^2-\{0\}$ is an orbit of the standard $SL_2(\mathbb{C})$ action.

5. Axiomatic characterization of motivic Chern classes

5.1. **Notations.** Let a connected linear group G act on the quasiprojective variety V, and let $\mathbb{T}$ be the maximal torus of G. Let $\Omega\subset V$ be an orbit, and let $x_\Omega\in\Omega$. We consider the stabilizer subgroup G_{x_Ω}. We will use the shorthand notation $G_\Omega = G_{x_\Omega}$. This subgroup is defined up to conjugation. We have

$$K^G(\Omega)\simeq K^G(G/G_\Omega)\simeq K^{G_\Omega}(x_\Omega)\simeq \mathrm{R}(G_\Omega).$$

Let $\mathbb{T}_\Omega$ be the maximal torus in G_Ω. We may assume that $\mathbb{T}_\Omega\subset\mathbb{T}$. For a connected linear group G the restriction homomorphism to the maximal torus $\mathbb{T}$ is an embedding and we identify $\mathrm{R}(G)$ with the invariants $\mathrm{R}(\mathbb{T})^W$, where W is the Weyl group. Let us introduce the maps

$$\phi_\Omega : K^G(V)\to \mathrm{R}(\mathbb{T}_\Omega)$$

by the composition of the restriction homomorphisms and natural isomorphisms

$$K^G(V) \to K^G(\Omega) \simeq K^{G_\Omega}(pt) \to K^{\mathbb{T}_\Omega}(pt) \simeq \mathrm{R}(\mathbb{T}_\Omega)\,.$$

The homomorphism ϕ_Ω does not depend on the choices made.

The tangent space to Ω at x_Ω will be denoted by $T(\Omega)$. The group G_Ω acts on $T(\Omega)$. Consider a *normal slice* S_Ω of Ω at x_Ω, that is a $\mathbb{T}_\Omega$-invariant smooth subvariety transverse to Ω at x_Ω of complementary dimension. To construct a slice one can assume by [Sum] that V is a subvariety of a projective space $\mathbb{P}^N$ with a linear action of $\mathbb{T}_\Omega$. Further we can assume that x lies in an invariant affine chart of $\mathbb{P}^N$, moreover that $x_\Omega = 0 \in \mathbb{C}^N$. There exists a linear subspace in $W \subset \mathbb{C}^N$, which is transverse to V at 0 and is $\mathbb{T}_\Omega$-invariant. That is so, since we can find a decomposition $\mathbb{C}^N = T(\Omega) \oplus W$ equivariantly with respect to the action of $\mathbb{T}^0_x$. The slice is defined as $S_\Omega = V \cap W$. The tangent space $T_0 S_\Omega$ will be denoted by $\nu(\Omega)$ and called the normal space to the orbit. As a representation of $\mathbb{T}_\Omega$ it does not depend on the choices made.

We recall that for a representation $W \in \mathrm{R}(\mathbb{T})$ we set $\lambda_y(W) = \sum_{k=0}^{\dim W}[\Lambda^k W]y^k \in \mathrm{R}(\mathbb{T})[y]$ and hence $\lambda_{-1}(W) = \sum_{k=0}^{\dim W}(-1)^k[\Lambda^k W] \in \mathrm{R}(\mathbb{T})$.

Lemma 5.1. *Let Θ, Ω be orbits. We have*

$$\phi_\Theta(\mathrm{mC}(\Omega \subset V)) = \lambda_y(T(\Theta)^*) \cdot \mathrm{mC}(\Omega \cap S_\Theta \subset S_\Theta)|\{x_\Theta\} \in \mathrm{R}(\mathbb{T}_\Theta)[y]. \tag{14}$$

Proof. Suppose S_Θ is a transverse slice to the orbit Θ at the point $x = x_\Omega \in \Theta$. Let the map $f : X \to V$ be a resolution of the pair $(\overline{\Omega}, \partial\Omega)$, where $\partial\Omega = \overline{\Omega} \setminus \Omega$, satisfying the condition of §2.4, i.e. $f^{-1}(\partial\Omega) = D = \bigcup_{i=1}^s D_i$ is a smooth divisor with normal crossings. Moreover assume that the resolution is invariant with respect to automorphisms of V. In particular it is invariant with respect to the action of the group G. Such a resolution is obtained by application of the Bierston-Milman [BiMi] algorithm. The slice S_Θ is automatically transverse to each orbit in a neighborhood of x. For each $I \subset \underline{s}$ the image of the intersection $D_I = \bigcap_{i\in I} D_i$ is a sum of orbits. Precisely: if $y \in f(D_I)$, then $g \cdot y \in f(D_I)$ for every $g \in G$ since the action of G lifts to X preserving D_I. Moreover it follows that the map f restricted to D_I is transverse to S_Θ. Therefore $f^{-1}(S_\Theta) \cap D$ is a smooth normal crossing divisor in $f^{-1}(S_\Theta)$ and f restricted to $f^{-1}(S_\Theta)$ can be used to compute $\mathrm{mC}(\Omega \cap S_\Theta \subset S_\Theta)$. After restriction to the point $x = x_\Theta$, for each $I \subset \underline{s}$ we have

$$\phi_\Theta(f_*\lambda_y(T^*D_I))|x = \lambda_y(T^*_x\Theta) \cdot f_*\lambda_y(T^*(D_I \cap f^{-1}(S_\Theta)))|x \in \mathrm{R}(\mathbb{T}_\Theta)[y].$$

Using the formula (2) of §2.4 we obtain the claim. □

Remark 5.2. The restriction formula holds in a more general situation, not necessarily when we deal with a space with a group action, stratified by orbits. It is a version of Verdier-Riemann-Roch. For CSM-classes an analogous formula was discussed by Yokura who gave a proof for smooth morphism [Yo2, Th. 2.2]. His argument applied to the natural map $G\times^{\mathbb{T}_\Theta} S_\Theta \to V$ is valid for equivariant motivic Chern classes as well. The proof of Verdier-Riemann-Roch by Schurmann [Sch1, Cor 0.1] also works in equivariant K-theory, provided that we apply the specialization functor in the category of equivariant mixed Hodge modules. Another proof for CSM-classes was given by Ohmoto [O2, Prop. 3.8]. In an earlier version of this paper the proof was attained is a similar way as ours. Nevertheless the exact statement and a proof for equivariant K-theory is not accessible in the literature, therefore we gave it here.

Theorem 5.3. *Let V be a smooth variety (not necessarily complete) on which an algebraic group G acts with finitely many orbits. Let us assume that*
(*) for all Ω the action of $\mathbb{T}_\Omega$ on $\nu(\Omega)$ is positive (see Definition 4.4).
Then the following properties hold
(i) $\phi_\Omega(\mathrm{mC}(\Omega \subset V)) = \lambda_y(T(\Omega)^*) \cdot \lambda_{-1}(\nu(\Omega)^*) \in \mathrm{R}(\mathbb{T}_\Omega)[y]$,
(ii) $\phi_\Theta(\mathrm{mC}(\Omega \subset V))$ *is divisible by* $\lambda_y(T(\Theta)^*)$ *in* $\mathrm{R}(\mathbb{T}_\Theta)[y]$,
(iii) if $\Theta \neq \Omega$ *then* $\mathcal{N}(\phi_\Theta(\mathrm{mC}(\Omega \subset V))/\lambda_y(T(\Theta)^*) \subset \mathcal{N}(\lambda_{-1}(\nu(\Theta)^*)) \setminus \{0\}$.

Remark 5.4. Observe that using (4), (5), and the fact that $0 \in \mathcal{N}(\lambda_y(T(\Theta)^*))$ condition (iii) is equivalent to
(iv) *if* $\Theta \neq \Omega$ *then* $\mathcal{N}(\phi_\Theta(\mathrm{mC}(\Omega \subset V))) \subset \mathcal{N}(\lambda_y(T(\Theta)^*)\lambda_{-1}(\nu(\Theta)^*)) \setminus \{0\}$,
or, what is the same, to
(v) *if* $\Theta \neq \Omega$ *then* $\mathcal{N}(\phi_\Theta(\mathrm{mC}(\Omega \subset V))) \subset \mathcal{N}(\phi_\Theta(\mathrm{mC}(\Theta \subset V))) \setminus \{0\}$.

Proof. (i) Since Ω is smooth
$$\mathrm{mC}(\mathrm{id}_\Omega)) = \lambda_y(T(\Omega)^*) \in K^G(\Omega)[y].$$
The composition of inclusion into V and restriction introduces the factor $\lambda_{-1}(\nu(\Omega)^*)$, the K-theoretic Euler class of $\nu(\Omega)$.
(ii) Follows from Lemma 5.1.
(iii) By Corollary 4.5 we have
$$\mathcal{N}(\mathrm{mC}(\Omega \cap S_\Theta \subset S_\Theta)|x_\Theta) \subset \mathcal{N}(\lambda_{-1}(\nu(\Theta)^*)) \setminus \{0\}. \tag{15}$$
Applying the multiplicative property (5) of Newton polygons and formula (14) we obtain the claim. ☐

Theorem 5.5. *Suppose that V is a smooth algebraic G-variety which is the union of finitely many orbits. Assume that the stabilizers of the orbits are connected and the action satisfies the positivity condition (*). Then the properties (i)-(iii) of Theorem 5.3 determine* $\mathrm{mC}(\Omega \subset V) \in K^G(V)$.

Our proof is an adaptation of the arguments in [O, Prop. 9.2.2], [RTV2, Section 3.2].

Proof. Let us fix a linear order of orbits such that
$$V_{\succ\Theta} = \bigsqcup_{\Omega \succ \Theta} \Theta$$
is an open set in V. Let $V_{\succeq\Theta} = \bigsqcup_{\Omega \succeq \Theta} \Theta$. First note that since the orbits have even real dimensions, thus the long exact sequences for the pairs $(V_{\succeq\Theta}, V_{\succ\Theta})$ split into short exact sequences
$$0 \longrightarrow K^G(\Theta) \xrightarrow{(\iota_\Theta)_*} K^G(V_{\succeq\Theta}) \longrightarrow K^G(V_{\succ\Theta}) \longrightarrow 0, \tag{16}$$
see §9 for details. Since the stabilizers of the orbits are connected we have
$$K^G(\Theta) = K^{G_\Theta}(pt) = \mathrm{R}(\mathbb{T}_\Theta)^{W_\Theta}.$$

Suppose that $\tau(\Omega)$ satisfies the conditions (i)-(iii). Then the difference $\delta = \tau(\Omega) - \mathrm{mC}(\Omega)$ satisfies

(i') $\phi_\Omega(\delta) = 0 \in \mathrm{R}(\mathbb{T}_\Omega)[y]$,

(ii') $\phi_\Theta(\delta)$ is divisible by $\lambda_y(T^*\Theta)$ in $\mathrm{R}(\mathbb{T}_\Theta)[y]$

(iii') $\mathcal{N}(\delta) \subsetneq \mathcal{N}(\lambda_y(T^*\Theta)\cdot\lambda_{-1}(\nu^*_\Theta))$ for all orbits, including $\Theta=\Omega$.

The condition (iii') holds becouse under the positivity assumption (*) the point 0 is a vertex of $\lambda_{-1}(\nu^*_\Theta)$, hence it does not belong to

$$\mathcal{N}(\phi_\Theta(\delta)/\lambda_y(T^*\Theta)) \subset conv.hull(\mathcal{N}(\phi_\Theta(\tau))/\lambda_y(T(\Theta)^*)) \cup \mathcal{N}(\phi_\Theta(\mathrm{mC}(\Omega)))/\lambda_y(T(\Theta)^*)))\,.$$

We argue by the induction on orbits that

$$\delta|V_{\succeq\Theta} = 0. \tag{17}$$

If Θ is the open orbit, then $V_{\succeq\Theta} = \Theta$ and $\lambda_{-1}\nu(\Theta) = 1$. The property (iii') implies that $\mathcal{N}(\phi_\Theta(\delta)) = \emptyset$, thus $\phi_\Theta(\delta) = 0 \in \mathrm{R}(\mathbb{T}_\Theta)[y]$. Since G_Θ is connected the restriction map $\mathrm{R}(G_\Theta) \to \mathrm{R}(\mathbb{T}_\Theta)$ is injective. Hence $\delta|\Theta = 0$.

Suppose by the inductive assumption that $\delta|V_{\succ\Theta} = 0$. We prove that $\delta|V_{\succeq\Theta} = 0$. By the exact sequence (16) the class $\delta|V_{\succeq\Theta}$ is the image of an element

$$\gamma \in K^G(\Theta)[y] \hookrightarrow K^{\mathbb{T}_\Theta}(pt)[y]\,.$$

We have

$$\iota^*_\Theta(\iota_\Theta)_*(\gamma) = \lambda_{-1}(\nu^*_\Theta)\cdot\gamma\,.$$

On the other hand $\iota^*_\Theta(\iota_\Theta)_*(\gamma)$ is equal to the restriction of δ to Θ. By (ii') it is divisible by $\lambda_y(T^*\Theta)$ in $\mathrm{R}(\mathbb{T}_\Theta)[y]$. The Laurent polynomials $\lambda_y(T^*\Theta)$ and $\lambda_{-1}(\nu^*_\Theta)$ are coprime in $\mathrm{R}(\mathbb{T}_\Theta)[y]$. Thus $\phi_\Theta(\delta)$ is divisible by $\lambda_y(T^*\Theta)\cdot\lambda_{-1}(\nu^*_\Theta)$. This contradicts the proper inclusion of Newton polygons. The only possibility is that $\phi_\Theta(\delta) = 0$, and as in the initial step we conclude that $\delta|\Theta = 0$. From the exactness of the sequence (16) it follows that $\delta|V_{\succeq\Theta} = 0$. □

The Newton polygon containment axiom, Theorem 5.3 (iii), says that a convex polygon is contained in another one minus a vertex, the origin. A stronger Newton polygon containment property is considered in [O, Section 9.1], which essentially says that the small convex polygon remains inside the larger one even if shifted slightly towards the origin (at least for a specific choice of slope parameter [O, 9.1.9]). The classes characterized by Okounkov's axiom are called K-theoretic stable envelopes, and they are only defined for very special varieties. Hence, a corollary of Theorems 5.3, 5.5 is that if a stable envelope is defined, it must coincide with the motivic Chern class. For general varieties only the weak Newton polygon containment holds, see e.g. Example 4.6.

5.2. **On the conditions of Theorem 5.5.** Theorem 5.5 provides an axiomatic characterization of $\mathrm{mC}(\Omega\subset V)$ if certain conditions hold. One of the conditions is that the stabilizer subgroups of the orbits are connected. The following examples show that this condition is indeed required.

Example 5.6. Let the torus $\mathbb{T}$ act on $V = \mathbb{T}/H$ where H is a finite subgroup of $\mathbb{T}$. The action has a unique orbit Ω. We have $K^{\mathbb{T}}(V)[y] = Rep(H)[y]$. Since the maximal torus $\mathbb{T}_\Omega$ of the stabilizer G_Ω is trivial, we have $Rep(\mathbb{T}_\Omega)[y] = \mathbb{Z}[y]$, and hence the map ϕ_Ω is the obvious forgetful map $Rep(H)[y] \to \mathbb{Z}[y]$. Since this map in not injective (not even after tensoring with $\mathbb{Q}$!) the uniqueness statement in Theorem 5.5 does not hold. On the other hand the values of ϕ_Ω and of the Chern character are determined by (i)-(iii), since here the target groups do not contain much information.

Example 5.7. Let $X = \mathbb{C}$, $G = \mathbb{T} = \mathbb{C}^*$ acting on $\mathbb{C}$ by $t \cdot z = t^n z$. Let $\Omega = \mathbb{C} \setminus \{0\}$, $\Theta = \{0\}$. We have a short exact sequence of $\mathrm{R}(\mathbb{C}^*)$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\mathbb{C}^*}(\Theta) & \xrightarrow{(\iota_\Theta)_*} & K^{\mathbb{C}^*}(\mathbb{C}) & \xrightarrow{\iota_\Omega^*} & K^{\mathbb{C}^*}(\Omega) & \longrightarrow & 0\,. \\ & & \| & \xleftarrow{\iota_\Theta^*} & & & \| & & \\ & & \mathrm{R}(\mathbb{C}^*) & & & & \mathrm{R}(\mathbb{Z}/(n)) & & \end{array}$$

We have $\mathbb{T}_\Omega = 1$, $\mathbb{T}_\Theta = \mathbb{T}$ and the restriction $\phi_\Theta = \iota_\Theta^* : K^{\mathbb{T}}(\mathbb{C}) \to K^{\mathbb{T}}(\Theta)$ is an isomorphism. Naming the generator of $\mathrm{R}(\mathbb{C}^*)$ by α we obtain that the composition $\iota_\Theta^*(\iota_\Theta)_*$ is the multiplication by $1 - \alpha^{-n}$, and the sequence

$$0 \longrightarrow \mathbb{Z}[\alpha, \alpha^{-1}] \xrightarrow{(1-\alpha^{-n})\cdot} \mathbb{Z}[\alpha, \alpha^{-1}] \longrightarrow \mathbb{Z}[\alpha]/(\alpha^n - 1) \longrightarrow 0\,.$$

Suppose $\tau \in K^{\mathbb{T}}(\mathbb{C})$ satisfies (i)–(iii). Property (i) fixes the value $\phi_\Omega(\tau) = 1 + y \in K^{\{1\}}(x_\Omega)[y] = \mathbb{Z}[y]$. Property (ii) says that $\phi_\Theta(\tau)$ is divisible by 1. Property (iii) gives the inclusion of Newton polygons $\mathcal{N}(\phi_\Theta(\tau)) \subset [-n, 1]$.

Therefore the classes of the form $(1+y)\alpha^{-k}$ with any $k \in [1, n]$ satisfy (i)-(iii). Hence for $n > 1$ the class τ is not determined by (i)-(iii) in $K^{\mathbb{T}}(\mathbb{C})$, nor the values of $\phi_\Theta(\tau)$, nor the value of the Chern character.

We believe that a modification of Theorem 5.5 should be true with *non-necessarily connected stabilizers*, but some condition on the discrete part of $\mathrm{mC}(\Omega \subset V)|\Theta$ should be imposed to guarantee uniqueness. Moreover, we believe that there is a version of Theorem 5.5 dealing with *non-necessarily positive* actions. We plan to study these extensions in the future.

6. Motivic Chern classes of Schubert cells in partial flag varieties

In this section we reinterpret results of [RTV3] to our settings, and thus we obtain explicit rational function representatives of motivic Chern classes of Schubert cells in partial flag varieties.

6.1. **The partial flag variety and its Schubert cells.** Let N, n be non-negative integers, and let $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{N}^N$, such that $\sum_{i=1}^N \mu_i = n$. Define $\mu^{(j)} = \sum_{i=1}^j \mu_i$. Consider the flag variety Fl_μ parameterizing flags of subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{N-1} \subset V_N = \mathbb{C}^n$$

with $\dim V_j = \mu^{(j)}$. The tautological bundle of rank $\mu^{(j)}$, whose fiber over $V_\bullet$ is V_j, will be denoted by $\mathcal{F}_j$.

Let $I=(I_1,...,I_N)$ be a partition of $\{1,...,n\}$ into disjoint subsets $I_1,...,I_N$ with $|I_j|=\mu_j$. The set of such I's will be denoted by $\mathcal{I}_\mu$. We will use the following notation: For $I\in\mathcal{I}_\mu$ let $I^{(j)}=\cup_{i=1}^j I_i$ and $I^{(j)}=\{i_1^{(j)}<\ldots<i_{\mu^{(j)}}^{(j)}\}$.

For $I\in\mathcal{I}_\mu$ define the Schubert cell

$$\Omega_I=\{V_\bullet\in \mathrm{Fl}_\mu : \dim(V_p\cap \mathbb{C}^q_{\mathrm{last}})=\#\{i\in I_1\cup\ldots,\cup I_p : i>n-q\},\forall p,q\},$$

where $\mathbb{C}^q_{\mathrm{last}}$ is the span of the last q standard basis vectors in $\mathbb{C}^n$. We have $\operatorname{codim}\Omega_I=\#\{(a,b)\in \underline{n}\times\underline{n} : a>b, a\in I_j, b\in I_k, j<k\}$.

6.2. **The equivariant K-ring of** Fl_μ**.** The standard action of the torus $\mathbb{T}=(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces an action on Fl_μ and the bundles $\mathcal{F}_j$. Denote the K-theoretic Chern roots of $\mathcal{F}_j$ by $\alpha_a^{(j)}$ $(a=1,\ldots,\mu^{(j)})$, that is $\sum_a\alpha_a^{(j)}=\mathcal{F}_j$ in $K^{\mathbb{T}}(\mathrm{Fl}_\mu)$. Observe that $\alpha_a^{(N)}$ are the Chern roots of the trivial $\mathbb{C}^n$ bundle with the standard $\mathbb{T}$-action, that is, the α-variables with upper index N are variables in $K^{\mathbb{T}}(\mathrm{pt})$. The algebra $K^{\mathbb{T}}(\mathrm{Fl}_\mu)$ is a certain quotient of the Laurent polynomial ring

$$\mathbb{Z}\left[(\alpha_a^{(j)})^{\pm1}\right]^{S_{\mu^{(1)}}\times\ldots\times S_{\mu^{(N-1)}}}_{j=1,\ldots,N,a=1,\ldots,\mu^{(j)}}, \tag{18}$$

by an ideal—not needed in this paper—whose generators express the fact that the bundles $\mathcal{F}_j\,/\,\mathcal{F}_{j-1}$ have rank μ_j.

Another way of describing the algebra $K^{\mathbb{T}}(\mathrm{Fl}_\mu)$ is equivariant localization. The torus fixed points of Fl_μ are flags of coordinate subspaces, they are also parameterized by $\mathcal{I}_\mu$. The restriction homomorphism $r_I : K^{\mathbb{T}}(\mathrm{Fl}_\mu)\to K^{\mathbb{T}}(x_I)$ to the fix point x_I corresponding to $I\in\mathcal{I}_\mu$ is the substitution

$$r_I:\alpha_a^{(j)}\mapsto\alpha^{(N)}_{i_a^{(j)}}\qquad\text{for } j=1,\ldots,N-1, a=1,\ldots,\mu^{(j)}. \tag{19}$$

Example 6.1. We have

$$K^{\mathbb{T}}(\mathrm{Fl}_{(1,1,1)})=\mathbb{Z}\left[\left(\alpha_1^{(1)}\right)^{\pm1};\left(\alpha_1^{(2)}\right)^{\pm1},\left(\alpha_2^{(2)}\right)^{\pm1};\left(\alpha_1^{(3)}\right)^{\pm1},\left(\alpha_2^{(3)}\right)^{\pm1},\left(\alpha_3^{(3)}\right)^{\pm1}\right]^{S_2}/\text{ideal},$$

and the restriction map

$$K^{\mathbb{T}}(\mathrm{Fl}_{(1,1,1)})\to K^{\mathbb{T}}(x_{(\{u\},\{v\},\{w\})})=\mathbb{Z}\left[\left(\alpha_1^{(3)}\right)^{\pm1},\left(\alpha_2^{(3)}\right)^{\pm1},\left(\alpha_3^{(3)}\right)^{\pm1}\right]$$

to $x_{(\{u\},\{v\},\{w\})}$, where (u,v,w) is a permutation of $(1,2,3)$, is induced by

$$\alpha_1^{(1)}\mapsto\alpha_u^{(3)},\qquad \alpha_1^{(2)}\mapsto\alpha^{(3)}_{\min(u,v)},\alpha_2^{(2)}\mapsto\alpha^{(3)}_{\max(u,v)}.$$

Because of the S_2-symmetry in $\alpha_1^{(2)},\alpha_2^{(2)}$, the same map is obtained by

$$\alpha_1^{(1)}\mapsto\alpha_u^{(3)},\qquad \alpha_1^{(2)}\mapsto\alpha_u^{(3)},\alpha_2^{(2)}\mapsto\alpha_v^{(3)}.$$

6.3. **Weight functions, modified weight functions.** For $I \in \mathcal{I}_\mu$, $j = 1, \dots, N-1$, $a = 1, \dots, \mu^{(j)}$, $b = 1, \dots, \mu^{(j+1)}$ define

$$\psi_{I,j,a,b}(\xi) = \begin{cases} 1-\xi & \text{if } i_b^{(j+1)} < i_a^{(j)} \\ (1+y)\xi & \text{if } i_b^{(j+1)} = i_a^{(j)} \\ 1+y\xi & \text{if } i_b^{(j+1)} > i_a^{(j)}. \end{cases}$$

Remark 6.2. It is worth comparing the values of this function with the fundamental calculation in Section 2.7.

Define the "weight function"

$$W_I = \mathrm{Sym}_{S_{\mu^{(1)}} \times \ldots \times S_{\mu^{(N-1)}}} U_I$$

where

$$U_I = \prod_{j=1}^{N-1} \prod_{a=1}^{\mu^{(j)}} \prod_{b=1}^{\mu^{(j+1)}} \psi_{I,j,a,b}(\alpha_a^{(j)}/\alpha_b^{(j+1)}) \cdot \prod_{j=1}^{N-1} \prod_{1 \le a < b \le \mu^{(j)}} \frac{1 + y\alpha_b^{(j)}/\alpha_a^{(j)}}{1 - \alpha_b^{(j)}/\alpha_a^{(j)}}.$$

Here the symmetrizing operator is defined by

$$\mathrm{Sym}_{S_{\mu^{(1)}} \times \ldots \times S_{\mu^{(N-1)}}} = \sum_{\sigma \in S_{\mu^{(1)}} \times \ldots \times S_{\mu^{(N-1)}}} U_I(\sigma(\alpha_a^{(j)}))$$

where the jth component of σ (an element of $S_{\mu^{(j)}}$) permutes the $\alpha^{(j)}$ variables. For

$$e_\mu = \prod_{j=1}^{N-1} \prod_{a=1}^{\mu^{(j)}} \prod_{b=1}^{\mu^{(j)}} (1 + y\alpha_b^{(j)}/\alpha_a^{(j)})$$

define the "modified weight function"

$$\widetilde{W}_I = W_I / e_\mu.$$

Observe that $\widetilde{W}_I$ is not a Laurent polynomial, but rather a ratio of two such.

Lemma 6.3. *The r_J-image (c.f. (19)) of $\widetilde{W}_I$ for any $J \in \mathcal{I}_\mu$ is a Laurent polynomial. There exists a* Laurent polynomial *in the ring (18) whose r_J-images are the same as those of $\widetilde{W}_I$ for all J. The class in $K^{\mathbb{T}}(\mathrm{Fl}_\mu)$ of this other Laurent polynomial will be denoted by $[\widetilde{W}_I]$.*

Proof. The statement is a special case of [RTV3, Lemma 3.3 and Section 5.2], see also Remark 6.5. □

Theorem 6.4. *We have*

$$\mathrm{mC}(\Omega_I \subset \mathrm{Fl}_\mu) = [\widetilde{W}_I] \in K^{\mathbb{T}}(\mathrm{Fl}_\mu).$$

Proof. The r_J-images of $[\widetilde{W}_I]$ satisfy the axioms for $\mathrm{mC}(\Omega_I \subset \mathrm{Fl}_\mu)$ in Theorems 5.3, 5.5, for the B_n^--action on Fl_μ. This statement is a special case of Lemma 3.5, Lemma 3.6, and Theorem 3.9 of [RTV3], see also Remark 6.5 □

Remark 6.5. In the proof of Lemma 6.3 and Theorem 6.4 we cited "special cases" of results in [RTV3]. Here let us explain to the reader what needs to be "specialized" in the results of [RTV3] for the purpose of our proofs. In [RTV3, Section 3.1] "trigonometric weight functions" $W^{\Delta}_{\sigma,I}$ and "modified trigonometric weight functions" $\widetilde{W}^{\Delta}_{\sigma,I}$ are defined, depending on three combinatorial parameters: $I \in \mathcal{I}_\mu$, $\sigma \in S_n$, and an "alcove" Δ. Our weight functions W_I and modified weight functions $\widetilde{W}_I$ only depend on the parameter $I \in \mathcal{I}_\mu$. The fact is that our weight functions are special cases of $W^{\Delta}_{\sigma,I}$ and $\widetilde{W}^{\Delta}_{\sigma,I}$ for special choices of σ and Δ. The choice of σ is $\sigma = \mathrm{id}$. The alcove Δ is characterized by a sequence of integers $m_{i,j}$ in [RTV3]. The specialization we need for our paper is $m_{i,j} = -1$ for all i, j. For these choices, the cited Lemmas and Theorem of [RTV3] prove axioms (i), (ii), and a property stronger than (iii) of Theorem 5.3.

Remark 6.6. In light of Remark 6.5 it is natural to ask how to modify the notion "motivic Chern class of the Schubert cell Ω_I" so that it equals the formula $\widetilde{W}^{\Delta}_{\sigma,I}$ of [RTV3], not just its special case $\widetilde{W}^{(m_{i,j}=-1)}_{\mathrm{id},I}$. The role of $\sigma \in S_n$ is simple, it corresponds to choosing a different reference full flag when defining the Schubert cells. We plan to explore the role of the alcove Δ (also called dynamical parameters, or *slope* in works of Okounkov), in future. In the previous works [RTV2] the alcove $(m_{i,j} = 0)$ was studied. The weight function $\widetilde{W}^{(m_{i,j}=0)}_{\mathrm{id},I}$ differs from $\widetilde{W}^{(m_{i,j}=-1)}_{\mathrm{id},I}$ by the Verdier duality [Sch2, Cor. 5.19] and a normalizing factor depending on the dimension of the cell.

Example 6.7. Consider $\mu = (2,2)$ and hence $\mathrm{Fl}_\mu = \mathrm{Gr}_2\,\mathbb{C}^4$. For the motivic Chern class of the Schubert cell corresponding to $(\{1,3\},\{2,4\}) \in \mathcal{I}_{(2,2)}$ we get

$$\mathrm{mC}(\Omega_{\{1,3\},\{2,4\}} \subset \mathrm{Gr}_2\,\mathbb{C}^4) = \left[\frac{1}{(1+y)^2(1+\frac{y\alpha_2}{\alpha_1})(1+\frac{y\alpha_1}{\alpha_2})}\,(U_1+U_2)\right]$$

where

$$U_1 = \frac{(1+y)^2\frac{\alpha_1\alpha_2}{\beta_1\beta_3}(1+\frac{y\alpha_1}{\beta_2})(1+\frac{y\alpha_1}{\beta_3})(1+\frac{y\alpha_1}{\beta_4})(1-\frac{\alpha_2}{\beta_1})(1-\frac{\alpha_2}{\beta_2})(1+\frac{y\alpha_2}{\beta_4})(1+\frac{y\alpha_2}{\alpha_1})}{(1-\frac{\alpha_2}{\alpha_1})}$$

and $U_2(\alpha_1,\alpha_2) = U_1(\alpha_2,\alpha_1)$. Here we used the short-hand notation $\alpha_i = \alpha_{1,i}$, $\beta_i = \alpha_{2,i}$. As seen, we represented $\mathrm{mC}(\Omega_{\{1,3\},\{2,4\}})$ with a rational function. Yet, its restrictions to all Schubert cells (or, equivalently, torus fixed points) are Laurent polynomials. For example the restriction to $\Omega_{\{1,2\},\{3,4\}}$ is 0, the restriction to $\Omega_{\{1,3\},\{2,4\}}$ is $(1+y\beta_1/\beta_2)\,(1+y\beta_1/\beta_4)\,(1-\beta_3/\beta_2)\,(1+y\beta_3/\beta_4)$, and the restriction to $\Omega_{\{3,4\},\{1,2\}}$ is

$$\frac{(1+y)\beta_4}{\beta_1^2\beta_2^2}\Big(y^2(\beta_1\beta_2\beta_3-\beta_3^2\beta_4) + y(2\beta_1\beta_2\beta_3+\beta_1\beta_2\beta_4-\beta_1\beta_3\beta_4+\beta_2^2\beta_3-\beta_2\beta_3^2-2\beta_2\beta_3\beta_4)+\beta_1\beta_2^2-\beta_2\beta_3\beta_4\Big).$$

According to the Newton polygon axiom Theorem 5.3(iii), the Newton polygon of this last expression needs to be contained in the Newton polygon of $(1-\beta_3/\beta_1)(1-\beta_3/\beta_2)(1-\beta_4/\beta_1)(1-\beta_4/\beta_2)$ (minus the origin); this containment is illustrated in Figure 2.

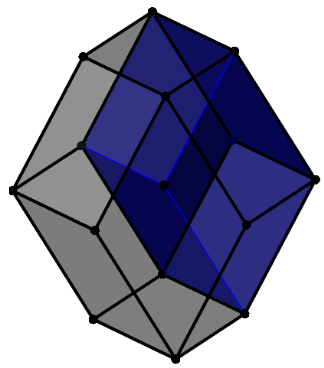


FIGURE 2. Newton polygons of Example 6.7

It is natural to think that Theorem 6.4 has a geometric proof, which, for example, explains Remark 6.2. In fact, this is indeed the case, one may use the traditional resolution method to achieve Theorem 6.4. Although such a proof—based on the fact that flag varieties can be obtained as GIT (or symplectic) quotients of vector spaces—has advantages, we decided not to follow that line of reasoning for two reasons. On the one hand a proof already exists in [RTV3] as we cited. On the other hand, in the related case of "matrix Schubert cells" we carry out such a program in the next section.

7. Motivic Chern classes of matrix Schubert cells

For $k \leq n$ let $\mathrm{GL}_k(\mathbb{C}) \times B_n^-$ (B_n^- is the group of $n \times n$ lower triangular non-singular matrices) act on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $(A, B) \cdot M = BMA^{-1}$. The orbits of this action are parameterized by d-element subsets J of $\underline{n} = \{1, \dots, n\}$, where $0 \leq d \leq k$. We will use the notation $J = \{j_1 < j_2 < \dots < j_d\}$. The orbit corresponding to J is

$$\Omega_{k,n,J} = \{M \in \mathbb{C}^{n \times k} : \mathrm{rk}(\text{top } r \text{ rows of } M) = |J \cap \underline{r}|\}.$$

The motivic Chern classes of these orbits live in the $\mathrm{GL}_k(\mathbb{C}) \times B_n^-$-equivariant K-theory algebra of a point (extended by the formal variable y), i.e.

$$\mathbb{Z}[y, \alpha_1^{\pm 1}, \dots, \alpha_k^{\pm 1}, \beta_1^{\pm 1}, \dots, \beta_n^{\pm 1}]^{S_k}, \tag{20}$$

where the S_k-action permutes the α_u variables.

7.1. Weight functions for $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$.

Now we define another version of weight functions, also denoted by W. Since their indexing is different from weight functions of Section 6.3, the notational coincidence will not cause misunderstanding (c.f. Remark 7.2).

Definition 7.1. *Let $k \leq n$, $I = \{i_1 < \dots < i_d\} \subset \underline{n}$, $|I| = d \leq k$. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ be two sets of variables. Define*

$$W_{k,n,I}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{1}{(k-d)!} \sum_{\sigma \in S_k} U_{k,n,I}(\sigma(\boldsymbol{\alpha}), \boldsymbol{\beta})$$

where

$$U_{k,n,I}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{u=1}^{k}\prod_{v=1}^{n}\psi_{I,u,v}(\alpha_u/\beta_v)\cdot\prod_{u=1}^{d}\prod_{v=u+1}^{k}\frac{1+y\frac{\alpha_v}{\alpha_u}}{1-\frac{\alpha_v}{\alpha_u}}, \tag{21}$$

and

$$\psi_{I,u,v}(\xi) = \begin{cases} 1-\xi & \textit{if} \quad u>d \quad \textit{or} \quad v<i_u\,, \\ (1+y)\xi & \textit{if} \quad u\le d \quad \textit{and} \quad v=i_u\,, \\ 1+y\xi & \textit{if} \quad u\le d \quad \textit{and} \quad v>i_u\,. \end{cases}$$

Although formally the weight function looks a rational function, it is in fact an element of the Laurent polynomial ring (20).

Remark 7.2. The weight function of this section *in the special case of* $d=k$ is a special case of the notion of weight function in Section 6.3. Namely $W_{(I,\underline{n}-I)}(\boldsymbol{\alpha},\boldsymbol{\beta})$ (in the sense of Section 6.3) equals $W_{k,n,I}$ (in the sense of this section).

Example 7.3. We have

$$W_{1,2,\{1\}} = (1+y)\frac{\alpha_1}{\beta_1}\left(1+\frac{y\alpha_1}{\beta_2}\right), \qquad W_{1,2,\{2\}} = (1+y)\left(1-\frac{\alpha_1}{\beta_1}\right)\frac{\alpha_1}{\beta_2},$$

$$W_{1,2,\{\}} = \left(1-\frac{\alpha_1}{\beta_1}\right)\left(1-\frac{\alpha_1}{\beta_2}\right).$$

More generally

$$W_{1,n,\{u\}} = (1+y)\prod_{i=1}^{u-1}\left(1-\frac{\alpha_1}{\beta_i}\right)\cdot\frac{\alpha_1}{\beta_u}\cdot\prod_{i=u+1}^{n}\left(1+\frac{y\alpha_1}{\beta_i}\right), \qquad W_{1,n,\{\}} = \prod_{i=1}^{n}\left(1+\frac{y\alpha_1}{\beta_i}\right).$$

For larger k the expanded form of weight functions is less manageable, e.g.

$$W_{2,2,\{1,2\}} = (1+y)^2\frac{\alpha_1\alpha_2}{\beta_1\beta_2}\cdot\left(y^2\frac{\alpha_1\alpha_2}{\beta_1\beta_2}+y\left(-\frac{\alpha_1\alpha_2}{\beta_1\beta_2}+\frac{\alpha_1}{\beta_1}+\frac{\alpha_1}{\beta_2}+\frac{\alpha_2}{\beta_1}+\frac{\alpha_2}{\beta_2}-1\right)+1\right)$$

$$W_{2,4,\{2,3\}} = (1+y)^2\prod_{i=1}^{2}\left(1-\frac{\alpha_i}{\beta_1}\right)\left(1+y\frac{\alpha_i}{\beta_4}\right)\cdot\alpha_1\alpha_2 \times\left(\frac{(1-y)}{\beta_2\beta_3}+(y^2-y)\frac{\alpha_1\alpha_2}{\beta_2^2\beta_3^2}+y(\alpha_1+\alpha_2)\left(\frac{1}{\beta_2\beta_3^2}+\frac{1}{\beta_2^2\beta_3}\right)\right).$$

The reader may find it instructive to verify that the sum of all weight functions for $k=1$ and fixed n factors as $\prod_{i=1}^{n}(1+y\alpha_1/\beta_i)$. The analogous fact for all k,n will follow from Theorem 7.4.

7.2. **Motivic Chern classes of matrix Schubert cells.** Recall from the beginning of Section 7 the $\mathrm{GL}_k(\mathbb{C}) \times B_n^-$-action on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and its orbits $\Omega_{k,n,J}$.

Theorem 7.4. *For the* $\mathrm{GL}_k(\mathbb{C}) \times B_n^-$*-equivariant motivic Chern classes we have*

$$\mathrm{mC}(\Omega_{k,n,J} \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)) = W_{k,n,J}.$$

A key difference between this theorem and Theorem 6.4 is that here the named class lives in a (Laurent) polynomial ring, rather than in a quotient of such a ring by an ideal. Hence here the weigh function *is* the motivic Chern class, not just *represents* a motivic Chern class.

Proof. As before we have $J = \{j_1 < \ldots < j_d\} \subset \underline{n}$ with $|J| = d$. Let $\mathrm{Fl}_{d,k}$ be the partial flag variety parameterizing chains of subspaces

$$V^\bullet = (V^0 \supset V^1 \supset \cdots \supset V^d), \qquad \dim(V^i) = k - i,$$

let $M = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \times \mathrm{Fl}_{d,k}$, and define

$$\widetilde{\Omega}_{k,n,J} = \Big\{(f, V^\bullet) \in M \mid f(V^{u-1}) \subset F^{j_u - 1}, f(V^{u-1}) \not\subset F^{j_u} \text{ for all } u \in \underline{d} \text{ and } f(V^d) = \{0\}\Big\}.$$

Consider the $(\mathbb{C}^*)^{k+n}$-equivariant diagram

$$\begin{array}{ccccc}
\widetilde{\Omega}_{k,n,J} & \hookrightarrow & M & \xrightarrow{\pi_2} & \mathrm{Fl}_{d,k} \\
\downarrow \simeq & & \downarrow \rho & & \\
\Omega_{k,n,J} & \hookrightarrow & \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) & &
\end{array}$$

with the projections $\rho = \pi_1$ and π_2. The map ρ restricted to $\widetilde{\Omega}_{k,n,J}$ is an isomorphism to its image $\Omega_{k,n,J} \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ (this resolution is a version of the construction in [KL], and it also appeared in [K, FR2]). Therefore, by functoriality of mC classes, we have

$$\mathrm{mC}(\Omega_{k,n,J} \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)) = \rho_*(\mathrm{mC}(\widetilde{\Omega}_{k,n,J} \subset M)).$$

According to Lefschetz-Riemann-Roch [BFQ], for $\omega \in K^{\mathbb{T}}(M)$ we have

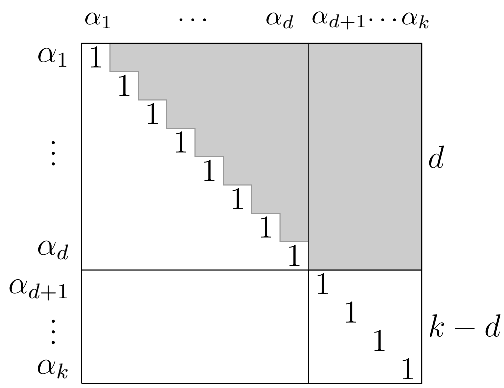
$$\frac{\rho_*(\omega)|0}{\lambda_{-1}(T_0^* \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n))} = \sum_{x \in M^{\mathbb{T}}} \frac{\omega|x}{\lambda_{-1}(T_x^* M)} \tag{22}$$

because the set of torus fixed points $M^{\mathbb{T}}$ is finite. The $\mathbb{T}$-fixed points of M are on the zero section $\mathrm{Fl}_{d,k}$, and they are parameterized by cosets S_k/S_{k-d}. We rewrite (22) for $\omega = \rho_*(\mathrm{mC}(\widetilde{\Omega}_{k,n,J} \subset M))$ as

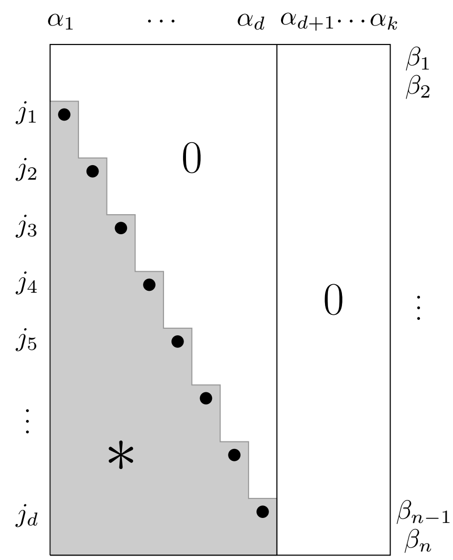
$$\rho_*(\mathrm{mC}(\widetilde{\Omega}_{k,n,J} \subset M))|0 = \sum_{x \in M^{\mathbb{T}}} \frac{\mathrm{mC}(\widetilde{\Omega}_{k,n,J} \subset M)|x}{\lambda_{-1}(T_x^* \mathrm{Fl}_{d,k})}. \tag{23}$$

One of the fixed points is $V_{\mathrm{last}} = (V^\bullet)$ with $V^u_{\mathrm{last}} = \mathrm{span}(\epsilon_{u+1}, \epsilon_{u+2}, \ldots, \epsilon_k)$ (with ϵ_i being the standard basis vectors of $\mathbb{C}^k$). First we calculate the term on the right hand side of (23) corresponding to this fixed point.

A neighborhood of $V^\bullet_{\rm last}$ in M, "vertically" is simply the neighborhood of $V^\bullet_{\rm last}$ in $\mathrm{Fl}_{d,k}$, which is naturally identified with the set of unipotent $k\times k$ matrices illustrated in the first picture below. The "horizontal" coordinates are the entries of an $n\times k$ matrix.



A neighborhood of $V^\bullet_{\rm last}$ in $\mathrm{Fl}_{d,k}$



Consider the subset of $n\times k$ matrices illustrated in the second figure, that is, the entries of the matrix labeled with 0 are 0, the entries labeled by $\bullet$ are non-zero complex numbers, and the entries labeled by $*$ are arbitrary complex numbers. Easy calculation from the definition of $\widetilde\Omega_{k,n,J}$ shows that the product of matrices should belong to the subset of $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$ described above. New coordinates are obtained by composing matrices (see Example 7.5 bellow).

Using the fundamental calculations of Section 2.7 we obtain that the numerator of the term corresponding to $V^\bullet_{\rm last}$ is the product of factors

- $1+y\alpha_v/\alpha_u$ for $1\le u\le d$, $u+1\le v\le k$ (coming from vertical directions, see the first picture);
- $1+y\alpha_u/\beta_v$ for $1\le u\le d$, $j_u<v\le n$ (the *-entries of the second picture);
- $1-\alpha_u/\beta_v$ for $1\le u\le d$ and $1\le v<j_u$, or $d<u\le k$, $1\le v\le n$ (the 0-entries of the second picture);
- $(1+y)\alpha_u/\beta_{j_u}$ for $1\le u\le d$ (the $\bullet$-entries in the second picture);

while the denominator is $\prod_{u=1}^d\prod_{v=1}^k(1-\alpha_v/\alpha_u)$.

That is, the term on the right hand side of (23) corresponding to the fixed point $V^\bullet_{\rm last}$ is exactly $U_{k,n,J}$ of (21). The terms corresponding to the other fixed points, i.e. corresponding to other cosets of S_k/S_{k-d}, are obtained by permuting the α-variables. Therefore the right hand side of (23) is $1/(k-d)!\cdot U_{k,n,J}(\sigma(\boldsymbol\alpha))=W_{k,n,J}$ what we wanted to prove. □

Example 7.5. Here we show the details of the coordinate change used in the proof above for the concrete example $k=4$, $n=4$, $J=\{2,3\}$ $(d=2)$. The neighbourhood of the standard flag

$V^\bullet_{\mathrm{last}}$, in the flag variety $\mathrm{Fl}_{2,4}$ is parameterized by the matrices of the form

$$\begin{pmatrix} \bullet & 0 & 0 & 0 \\ * & \bullet & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

stabilizer of $(V^1_{\mathrm{last}}, V^2_{\mathrm{last}})$, local coordinates.

The set $\widetilde{\Omega}_{2,4,\{2,3\}}$ is described by imposing conditions on the entries of the product of the matrices:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & 1 & g_{23} & g_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} f_{11} & f_{11}g_{12}+f_{12} & f_{11}g_{13}+f_{12}g_{23}+f_{13} & f_{11}g_{14}+f_{12}g_{24}+f_{14} \\ \boxed{\boldsymbol{f_{21}}} & f_{21}g_{12}+f_{22} & f_{21}g_{13}+f_{22}g_{23}+f_{23} & f_{21}g_{14}+f_{22}g_{24}+f_{24} \\ \boldsymbol{f_{31}} & \boxed{\boldsymbol{f_{31}g_{12}+f_{32}}} & f_{31}g_{13}+f_{32}g_{23}+f_{33} & f_{31}g_{14}+f_{32}g_{24}+f_{34} \\ \boldsymbol{f_{41}} & \boldsymbol{f_{41}g_{12}+f_{42}} & f_{41}g_{13}+f_{42}g_{23}+f_{43} & f_{41}g_{14}+f_{42}g_{24}+f_{44} \end{pmatrix}.$$

- the boxed entries should be nonzero,
- the bold entries are arbitrary,
- the remaining entries have to vanish.

Let us introduce new coordinates $\{f'_{st}\}_{s\in\underline{n},\, t\in\underline{k}}$, which coincide with the entries of the product matrix. Together with $\{g_{uv}\}_{1\le u\le d,\, u<v\le k}$ it is a coordinate system in a neighborhood of the fixed point $(0, V^\bullet_{\mathrm{last}}) \in M$, which lies in the closure of $\widetilde{\Omega}_{2,4,\{2,3\}}$. In the new coordinates the set $\widetilde{\Omega}_{2,4,\{2,3\}}$ is defined by the conditions:

$$\begin{array}{llll} f'_{11}=0, & f'_{12}=0, & f'_{13}=0, & f'_{14}=0, \\ \boxed{\boldsymbol{f'_{21}}\neq 0,} & f'_{22}=0, & f'_{23}=0, & f'_{24}=0, \\ & \boxed{\boldsymbol{f'_{32}}\neq 0}, & f'_{33}=0, & f'_{34}=0, \\ & & f'_{43}=0. & f'_{44}=0. \end{array}$$

Remark 7.6. A sketch of the cohomology version of the proof above appeared in [FR2, Sect. 12]. Not surprisingly the cohomological weight function (CSM-class) only differs from the one in this paper by carrying out the changes

$$1-\frac{1}{\alpha}\mapsto a, \qquad 1+\frac{y}{\alpha}\mapsto 1+a, \qquad \frac{1+y}{\alpha}\mapsto 1$$

for each of its factors, c.f. Remark 2.3.

8. A_2 quiver representation or determinantal varieties

8.1. **Segre version of motivic Chern class.** Imitating the way the Segre-Schwartz-MacPherson class is defined from the Chern-Schwartz-MacPherson class (namely SSM=CSM/total-Chern-class of ambient space), we define the Segre version of the motivic Chern class.

Definition 8.1. *For $X \to V$ (V smooth) define the motivic Segre class as*

$$\mathrm{mS}(X \to V) = \mathrm{mC}(X \to V)/\lambda_y(T^*V) \in K(V)[[y]].$$

We use the abbreviation $\mathrm{mS}(X)$ *if the map $X \to V$ is clear from the context, and we use the same notation for the equivariant version (living in $K^G(V)[[y]]$) if the map $X \to V$ is invariant under a group action.*

For an embedded smooth variety $\iota : X \hookrightarrow V$

$$\mathrm{mS}(\iota) = \frac{\iota_*\lambda_y(T^*X)}{\lambda_y(T^*V)} = \iota_*\left(\frac{\lambda_y(T^*X)}{\iota^*\lambda_y(T^*V)}\right) = \iota_*\left(\frac{1}{\lambda_y(\nu_X^*)}\right) = \frac{\lambda_{-1}(\nu^*)}{\lambda_y(\nu^*)}\,,$$

where ν is a vector bundle on V which is an extension of the normal bundle ν_X.

An important feature of both mC and mS is that they are not necessarily multiplicative with respect to fibrations. Nevertheless the formula

$$p_*(\mathrm{mC}(X \to Y)) = \chi_y(F)\,\mathrm{mC}(id_Y) \tag{24}$$

holds for fibrations $F \hookrightarrow X \overset{p}{\twoheadrightarrow} Y$ which are **locally trivial in Zariski topology**. This holds because for such fibration one can decompose the base into locally closed sets over which the bundle is of the product form.

8.2. **Fixed rank loci.** For $k \le n$ let $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ act on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $(A, B) \cdot M = BMA^{-1}$. The orbits of this action are characterized by rank, namely

$$\Sigma^r_{k,n} = \{\phi \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \dim\ker\phi = r\}$$

are the orbits for $r = 0, \dots, k$. We will be interested in the motivic Chern/Segre classes of these orbits living in

$$K^{\mathbb{T}}(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n))[[y]] = K^{\mathbb{T}}(\mathrm{pt})[[y]] = \mathbb{Z}[\alpha_1^{\pm1}, \dots, \alpha_k^{\pm1}, \beta_1^{\pm1}, \dots, \beta_n^{\pm1}][[y]]$$

where $\mathbb{T} = (\mathbb{C}^*)^k \times (\mathbb{C}^*)^n$, c.f. Section 7. The answers will be symmetric in the α_i and β_j variables separately (since the orbits are obviously $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$-invariant), and can be considered as the $\mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$-equivariant mC and mS classes.

In the next two sections we will present two formulas for mC (or mS) of $\Sigma^r_{k,n}$. The two expressions obtained in Theorems 8.2, 8.3 will be rather different, their equality is not clear algebraically.

8.3. **Motivic Chern classes of fixed rank loci — the motivic expression.** Recall the definition of weight functions $W_{k,n,J}$ from Section 7.1.

Theorem 8.2. *We have*

$$\mathrm{mC}(\Sigma^r_{k,n}) = \sum_{\substack{J\subset \underline{n}\\ |J|=k-r}} W_{k,n,J}(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

$$\mathrm{mS}(\Sigma^r_{k,n}) = \frac{1}{\prod_{i=1}^k\prod_{j=1}^n (1+y\alpha_i/\beta_j)}\cdot \sum_{\substack{J\subset \underline{n}\\ |J|=k-r}} W_{k,n,J}(\boldsymbol{\alpha},\boldsymbol{\beta}).$$

Proof. The set $\Sigma^r_{k,n}$ is the disjoint union of the sets $\Omega_{k,n,J}$ with $J\subset \underline{n}$, $|J|=k-r$. This decomposition is $\mathbb{T}=(\mathbb{C}^*)^k\times(\mathbb{C}^*)^n$-invariant. Hence the motivic properties of $\mathbb{T}$-equivariant mC and mS classes imply the statements. □

8.4. **Motivic Chern classes of fixed rank loci — the sieve expression.** In this section it will be convenient to denote $q=-y$. Let

$$\binom{a}{r}_q = \chi_{-q}(Gr_r(\mathbb{C}^a))\,.$$

be the q-binomial coefficient satisfying the recursion

$$\binom{a+1}{r}_q = q^r\binom{a}{r}_q + \binom{a}{r-1}_q, \qquad \binom{a}{0}_q = \binom{a}{a}_q = 1.$$

Theorem 8.3. *We have*

$$\mathrm{mS}(\Sigma^r_{k,n}) = \sum_{a=r}^{k} (-1)^{a-r} q^{\frac{(a-r)(a-r-1)}{2}} \binom{a}{r}_q \Phi^a_{k,n}, \tag{25}$$

where

$$\Phi^a_{k,n} = \sum_{I\in\binom{k}{a}} \prod_{u\in I}\prod_{v=1}^{n} \frac{1-\frac{\alpha_u}{\beta_v}}{1-q\frac{\alpha_u}{\beta_v}} \prod_{u\in I}\prod_{w\in I^\vee} \frac{1-q\frac{\alpha_u}{\alpha_w}}{1-\frac{\alpha_u}{\alpha_w}}. \tag{26}$$

(Here by $I\in\binom{k}{a}$ we mean that I is a subset of $\underline{k}$ of cardinality a, and $I^\vee$ is its complement.)

Proof. Let $\mathrm{Gr}_a(\mathbb{C}^k)$ denote the Grassmannian of a-dimensional subspaces in $\mathbb{C}^k$. Consider the smooth variety

$$\widetilde{\Sigma}^a_{k,n} = \{(f,W)\in V\times \mathrm{Gr}_a(\mathbb{C}^k)\mid W\subset \ker(f)\}$$

with its projection to V; and denote $\Phi^a_{k,n}=\mathrm{mS}(\widetilde{\Sigma}^a_{k,n}\to V)$. We claim that (26) holds. Indeed denoting by $p_a:\mathrm{Gr}_a(\mathbb{C}^k)\to \mathrm{pt}$ the projection we have

$$\Phi^a_{k,n} = p_{a*}\left(\frac{\lambda_{-1}(\nu^*)}{\lambda_{-q}(\nu^*)}\lambda_{-q}(T^*\mathrm{Gr}_a(\mathbb{C}^k))\right),$$

where $\nu = \mathrm{Hom}(\gamma_a, \mathbb{C}^n)$ and γ_a is the tautological bundle over $\mathrm{Gr}_a(\mathbb{C}^k)$. Calculating the push-forward map p_{a*} by localization in K-theory we arrive at (26).

By additivity and the Zariski-multiplicativity property (24) of mS we have

$$\Phi^r_{k,n} = \sum_{a=r}^{k} \binom{a}{r}_q \mathrm{mS}(\Sigma^a_{k,n}). \tag{27}$$

The elementary fact that the matrices

$$\left(\binom{a}{r}_q\right)_{1\le a,r\le k}, \qquad \left((-1)^{a-r} q^{\frac{(a-r)(a-r-1)}{2}} \binom{a}{r}_q\right)_{1\le a,r\le k}$$

are inverses to each other completes the proof of (25). □

Remark 8.4. It would be interesting to understand the k,n-dependence of the formulas in Theorem 8.3, either combinatorially, or via the language of iterated residues, cf. [FR2]. For now, what is clear from the formulas is that both $\Phi^r_{k,n}$ and $\mathrm{mS}(\Sigma^r_{k,n})$ are supersymmetric, i.e. satisfy the functional equation

$$f(\alpha_1,\dots,\alpha_k,t;\beta_1,\dots,\beta_n,t) = f(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_n),$$

with base case

$$\mathrm{mS}(\Sigma^k_{k,n}) = \mathrm{mS}(\{0\}\subset V) = \Phi^k_{k,n} = \prod_{u=1}^{k}\prod_{v=1}^{n} \frac{1-\frac{\alpha_u}{\beta_v}}{1-q\frac{\alpha_u}{\beta_v}}.$$

8.5. **Examples.** For two of the rank loci of the representation $\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2)$ we have

$$\mathrm{mS}(\Sigma^2_{2,2}) = \frac{\left(1-\frac{\alpha_1}{\beta_1}\right)\left(1-\frac{\alpha_2}{\beta_1}\right)\left(1-\frac{\alpha_1}{\beta_2}\right)\left(1-\frac{\alpha_2}{\beta_2}\right)}{\left(1-q\frac{\alpha_1}{\beta_1}\right)\left(1-q\frac{\alpha_2}{\beta_1}\right)\left(1-q\frac{\alpha_1}{\beta_2}\right)\left(1-q\frac{\alpha_2}{\beta_2}\right)},$$

$$\mathrm{mS}(\Sigma^0_{2,2}) = \frac{(q-1)^2}{\left(1-q\frac{\alpha_1}{\beta_1}\right)\left(1-q\frac{\alpha_2}{\beta_1}\right)\left(1-q\frac{\alpha_1}{\beta_2}\right)\left(1-q\frac{\alpha_2}{\beta_2}\right)} \cdot \frac{\alpha_1\alpha_2}{\beta_1\beta_2} \cdot \left(q^2\frac{\alpha_1\alpha_2}{\beta_1\beta_2} + q\left(\frac{\alpha_2\alpha_1}{\beta_1\beta_2} - \frac{\alpha_1}{\beta_1} - \frac{\alpha_1}{\beta_2} - \frac{\alpha_2}{\beta_1} - \frac{\alpha_2}{\beta_2} + 1\right) + 1\right).$$

The motivic Segre classes can be expressed in terms of double stable Grothendieck polynomials $G_\lambda = G_\lambda(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_n)$, see e.g. [RSz, Definition 4.2 and Theorem 4.5]. For example, for $k=n=2$ we have

$$\mathrm{mS}(\Sigma^0_{2,2}) = (G_0 - G_1) + q(-G_1 + G_2 + G_{11} - G_{21}) + q^2(-G_1 + 2G_2 - G_3 + 2G_{11} - 4G_{21} - G_{111} + 2G_{31} + 2G_{211} - G_{311}) + \dots$$

where experts on algebraic combinatorics can see the usual sign pattern of Grothendieck expansions (e.g. [Bu]). Further analysis along these lines is subject to future research.

9. Appendix: Basic information about topological K-theory

Let G be a compact group and let X be a compact G-space. By $K^G_{top}(X)$ we denote the Grothendieck ring of topological G-vector bundles over X. It is a contravariant functor on the category of G-spaces. According to Segal [Seg] this functor extends to a $\mathbb{Z}_2$ graded multiplicative generalized equivariant cohomology theory $K^*_G(-)$ defined for spaces which are of G-homotopy type of finite G-CW-complexes. It has the following properties:

(1) $K^0_G(X) = K^G_{top}(X)$.
(2) $K^0_G(G/H) \simeq \mathrm{R}(H)$, $K^1_G(G/H) = 0$.
(3) More generally $K^*_G(G \times_H X) \simeq K^*_H(X)$ for every H-space X, [Seg, §2(iii)].
(4) If the action of G on X is free, then $K^*_G(X) \simeq K^*(X/G)$, [Seg, Prop. 2.1].
(5) If the action of G on X is trivial, then $K^*_G(X) \simeq \mathrm{R}(G) \otimes K^*(X)$, [Seg, Prop. 2.2].
(6) (Thom isomorphism) There is a natural isomorphism of reduced K-theories: if $E \to X$ is an equivariant complex vector bundle, then

$$\widetilde{K}^*_G(Th(E)) \simeq \widetilde{K}^*_G(X)\,,$$

where $Th(E)$ is the Thom space of E, [Seg, Prop. 3.2]. It allows to define push-forward for proper maps of complex G-manifolds; this push forward agrees with the push-forward in algebraic K-theory for projective morphisms.
(7) If G is a connected Lie group with $\pi_1(G)$ torsion free, $\mathbb{T} \subset G$ the maximal torus, then

$$K^*_{\mathbb{T}}(X) \simeq \mathrm{R}(\mathbb{T}) \otimes_{\mathrm{R}(G)} K^*_G(X)\,,$$

see [BrZh, Lemma 2.5]. It follows that for such groups $K^*_G(X) \simeq K^*_{\mathbb{T}}(X)^W$, where W the Weyl group.
(8) The Borel construction leads to a map

$$\boldsymbol{\alpha} : K^*_G(X) \to K^*(EG \times_G X)\,,$$

where the K-theory of infinite CW-complexes are defined as homotopy classes of maps to the classifying space:

$$K^0(EG \times_G X) = [EG \times_G X\,,\, BU]\,, \qquad K^1(EG \times_G X) = [EG \times_G X\,,\, U]\,.$$

The map $\boldsymbol{\alpha}$ extends to an isomorphism of the I_G-adic completion

$$K^*_G(X)^\wedge_{I_G} \;\simeq\; K^*(EG \times_G X)\,,$$

where $I_G \subset K^0_G(pt) = \mathrm{R}(G)$ is the augumentation ideal

$$I_G = \ker(\dim : \mathrm{R}(G) \to \mathbb{Z})\,,$$

see [AS, Prop. 4.2].
(9) (Localization theorem.) Suppose that $G = \mathbb{T}$ is a torus. Then the kernel and the cokernel of the restriction to the fixed point set

$$K^*_{\mathbb{T}}(X) \to K^*_{\mathbb{T}}(X^{\mathbb{T}})$$

are torsion $\mathrm{R}(\mathbb{T})$-modules, [Seg, Prop. 4.1].

The Chern character of an equivariant vector bundle is obtained by applying the nonequivariant Chern character to the associated vector bundle on $EG \times_G X$. We obtain a map

$$ch_G : K^*_G(X) \to \hat{H}^*_G(X;\mathbb{Q}) = \prod_{k=0}^{\infty} H^k_G(X;\mathbb{Q})$$

This Chern character may have the kernel which is not $\mathbb{Z}$-torsion.

Example 9.1. If $G = S^1$, $H = \mathbb{Z}/(p)$, $X = G/H$ then we have:

- The equivariant K-theory
$$K^*_G(X) = \mathrm{R}(H) \simeq \mathbb{Z}[\mathbb{Z}/(p)]$$
is additively isomorphic to $\mathbb{Z}^p$.
- The K-theory of the Borel construction $K^*(EG \times_G X) \simeq K^*(BH)$ can be computed using the Atiyah-Segal completion theorem:
$$K^*(B\,\mathbb{Z}/(p)) \simeq \mathrm{R}(Z/(p))^{\wedge}_{I_{\mathbb{Z}/(p)}} \simeq \mathbb{Z} \oplus \mathbb{Z}^{\wedge}_p\,,$$
see [At, §8].
- Rational equivariant cohomology is trivial in positive degrees
$$\hat{H}^*(EG \times_G X;\mathbb{Q}) = \hat{H}^*(B\,\mathbb{Z}/(p);\mathbb{Q}) \simeq \mathbb{Q}\,.$$

If $G = S^1$, then $K^*_G(\mathrm{pt}) \simeq \mathbb{Z}[\xi,\xi^{-1}]$, $\hat{H}^*(BG;\mathbb{Q}) \simeq \mathbb{Q}[[x]]$ and the Chern character maps ξ to $\exp(x)$. In this case the map

$$K^*_G(\mathrm{pt}) \longrightarrow \hat{H}^*(BG;\mathbb{Q})$$

is injective.

For a connected group G with the maximal torus $\mathbb{T}$ and the Weyl group W we have a commutative diagram of injections

$$\begin{array}{ccccccccc}
\mathrm{R}(\mathbb{T})^W & = & \mathrm{R}(G) & = & K^*_G(pt) & \xhookrightarrow{ch_G} & \hat{H}^*(BG;\mathbb{Q}) & = & \hat{H}^*(BT;\mathbb{Q})^W \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & \mathrm{R}(\mathbb{T}) & = & K^*_{\mathbb{T}}(pt) & \xhookrightarrow{ch_{\mathbb{T}}} & \hat{H}^*(B\mathbb{T};\mathbb{Q}) & &
\end{array}$$

Assume that the variety X is smooth and projective, and an algebraic torus $\mathbb{T}$ acts with finitely many fixed points. Then due to Białynicki-Birula decomposition (for a generic 1-parameter subgroup) X is a sum of $\mathbb{T}$-equivariant algebraic cells. Then its equivariant K-theory has no $\mathrm{R}(\mathbb{T})$-torsion and by the localization theorem the restriction map $K^*_{\mathbb{T}}(X) \to K^*_{\mathbb{T}}(X^{\mathbb{T}})$ is an injection. Therefore for such tori actions we may check identities involving motivic Chern classes in equivariant cohomology of fixed points. On the other hand divisibility constrains are much less restrictive in equivariant cohomology.

When we deal with algebraic groups, then by $K^*_G(X)$ we mean the ring $K^*_{G_c}(X)$ where $G_c \subset G$ is the maximal compact subgroup. If H is an algebraic subgroup of G, then G/H is G_c-homotopy equivalent to G_c/H_c. The odd K-theory $K^1_{G_c}(G_c/H_c)$ vanish and $K^0_{G_c}(G_c/H_c) \simeq \mathrm{R}(H_c)$ is a free group. Therefore if the space X is a finite sum of orbits, then

$$K^*_G(X) \simeq \bigoplus_{[x]\in X/G} \mathrm{R}(G_x)\,.$$

This is an isomorphism of abelian groups, not necessarily $\mathrm{R}(G_c)$-modules. Similar situation appears for rational equivariant cohomology, [FW, Th. 1.7].

More generally, suppose X is smooth complex G-variety, with a filtration

$$\emptyset = U_0 \subset U_1 \subset U_2 \subset \dots \subset U_N = X$$

consisting of G-invariant open sets, such that $U_k \setminus U_{k-1}$ is a complex submanifold of U_k, which is G-homotopy equivalent to an orbit G/G_k for $k = 1, 2, \dots, N$. Then additively

$$K^*_G(X) \simeq \bigoplus_{k=1}^{N} \mathrm{R}(G_k)\,.$$

The proof is by inductive application of the exact sequence

$$0 \longrightarrow \mathrm{R}(G_k) \longrightarrow K^*_G(U_k) \longrightarrow K^*_G(U_{k-1}) \longrightarrow 0\,.$$

The results of this paper, in particular Theorem 5.3 and Theorem 5.5, with additional work apply to this general situation.

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