10
ON THE CONSTRUCTION FK

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1. Introduction

The reduced product construction of Ioan James [5] assigns to each CW-complex a new CW-complex having the same homotopy type as the loops in the suspension of the original. This paper will describe an analogous construction proceeding from the category of semi-simplicial complexes to the category of group complexes. The properties of this construction FK are studied in §2.

A theorem of Peter Hilton [4] asserts that the space of loops in a union $S_1 \vee \ldots \vee S_r$ of spheres splits into an infinite direct product of loops spaces of spheres. In §3 the construction of FK is applied to prove a generalization (Theorem 4) of Hilton's theorem in which the spheres may be replaced by the suspensions of arbitrary connected (semi-simplicial) complexes.

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2. The construction

It will be understood that with every semi-simplicial complex there is to be associated a specified base point.

Let K be a semi-simplicial complex with base point b_0. Denote $S_0^n b_0$ by b_n. Let FK_n denote the free group generated by the elements of K_n with the single relation $b_n = 1$. Let the face and degeneracy operations ∂_i, s_i in $FK = \cup FK_n$ be the unique homomorphisms which carry the generators k_n into $\partial_i k_n$, $s_i k_n$ respectively. Thus each complex K determines a group complex FK.

It will be shown that FK is a loop space for EK, the suspension of K. (Definitions will be given presently.)

Alternatively let $F^+K_n \subset FK_n$ be the free monoid (= associative semi-group with unit) generated by K_n, with the same relation $b_n = 1$. Then the monoid complex F^+K is also a loop space for EK. This construction is the direct generalization of James' construction. (See Lemma 4.)

The suspension EK of the semi-simplicial complex K is defined as follows. For each simplex k_n, other than b_n, of K there is to be a sequence (Ek_n), $(s_0 Ek_n)$, $(s_0^2 Ek_n)$, ... of simplexes of EK having dimensions $n+1$, $n+2$, ... In addition there is to be a base point (b_0) and its degeneracies (b_n). The symbols $(s_0^i Eb_n)$ will be identified with (b_{n+i+1}). The face and degeneracy operations in EK are given by

$$\partial_j(Ek_n) = (E\partial_{j-1}k_n) \qquad (j > 0,\ n > 0)$$

$$s_j(Ek_n) = (Es_{j-1}k_n) \qquad (j > 0)$$

$$\partial_0(Ek_n) = (b_n)\ , \qquad \partial_1(Ek_0) = (b_0)$$

$$s_0(Ek_n) = (s_0 Ek_n)\ .$$

The face and degeneracy operations on the remaining simplexes

$(s_0^i Ek_n) = s_0^i(Ek_n)$ are now determined by the identities

$$\partial_j s_0^i = \begin{cases} s_0^i \partial_{j-1} & (j > i) \\ s_0^{i-1} & (j \leq i \neq 0) \end{cases}$$

$$s_j s_0^i = \begin{cases} s_0^i s_{j-1} & (j > i) \\ s_0^{i+1} & (j \leq i) . \end{cases}$$

It is not hard to show that this defines a semi-simplicial complex. The following lemma will justify calling it the suspension of K. Recall that the suspension of a topological space A with base point a_0 is the identification space of $A \times I$ obtaining by collapsing $(A \times \dot{I}) \cup (a_0 \times I)$ to a point.

Lemma 1. The geometric realization $|EK|$ is canonically homeomorphic to the suspension of $|K|$.

(For the definition of realization see [6]. In fact the required homeomorphism is obtained by mapping the point $(|k_n, \delta_n|, 1-t)$ of the suspension of $|K|$, where δ_n has barycentric coordinates $(t_0, \ldots, t_n)$ into the point $|(Ek_n), \delta_{n+1}| \in |EK|$, where δ_{n+1} has barycentric coordinates $(1-t, tt_0, \ldots, tt_n)$.)

Next the space of loops on a semi-simplicial complex K will be discussed. If K satisfies the Kan extension condition then ΩK can be defined as in [7]. This definition has two disadvantages:

(1) Many interesting complexes do not satisfy the extension condition. In particular EK does not.

(2) There is no natural way (and in some cases† no possible way) of defining a group structure in ΩK.

The following will be more convenient. A group complex G, or more generally a monoid complex, will be called a loop space for K if there exists a (semi-simplicial) principal bundle with base space K, fibre G, and with contractible total space T.

(By a principal bundle is meant a projection p of T onto K together with a left translation $G \times T \to T$ satisfying

$$(g_n \cdot g'_n) \cdot t_n = g_n \cdot (g'_n \cdot t_n)$$

where $g_n \cdot t_n = t_n$ if and only if $g_n = 1_n$; and where $g_n \cdot t_n = t'_n$ for some g_n if and only if $p(t_n) = p(t'_n)$. A complex is called contractible if its geometric realization is contractible. This is equivalent to requiring that the integral homology groups and the fundamental group be trivial.)

The existence of such a loop space for any connected complex K has been shown in recent work of Kan, which generalizes the present paper. The following Lemma is given to help justify the definition.

Lemma 2. *If K satisfies the extension condition, and the group complex G is a loop space for K, then there is a homotopy equivalence $\Omega K \to G$.*

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† Let K be the minimal complex of the n-sphere $n \geq 2$. Then it can be shown that there is no group complex structure in ΩK having the correct Pontrjagin ring.

The proof is based on the following easily proven fact (compare [7] p. 2-10): Every principal bundle can be given the structure of a twisted cartesian product. That is one can find a one-one function

$$\eta : G \times K \to T$$

satisfying $\partial_i \eta = \eta \partial_i$ for $i > 0$ and $s_i \eta = \eta s_i$ for all i, where $\partial_0 \eta$ is given by an expression of the form

$$\partial_0 \eta(g_n k_n) = \eta((\partial_0 g_n) \cdot (\tau k_n), \; \partial_0 k_n) .$$

(For this assertion the fibre must be a monoid complex satisfying the extension condition.) Thus the bundle is completely described by G and K together with the 'twisting function' $\tau : K_n \to G_{n-1}$; where τ satisfies the identities

$$s_i \tau = \tau s_{i+1} \quad (i \geq 0), \qquad \partial_i \tau = \tau \partial_{i+1} \quad i \geq 1 ,$$

$$\tau s_0 k_n = 1_n , \qquad (\partial_0 \tau k_n) \cdot (\tau \partial_0 k_n) = \tau \partial_1 k_n .$$

Now a map $\bar{\tau} : \Omega K_{n-1} \to G_{n-1}$ is defined by $\bar{\tau}(k_n) = \tau(k_n)$. From the definition of ΩK and the above identities it follows that $\bar{\tau}$ is a map. From the homotopy sequence of the bundle it is easily verified that $\bar{\tau}$ induces isomorphisms of the homotopy groups, which proves Lemma 2.

To define a principal bundle with fibre FK and base space EK it is sufficient to define twisting functions $\tau : EK_{n+1} \to FK_n$. These will be given by

$$\tau(Ek_n) = k_n , \qquad \tau(s_0^i Ek_{n-i}) = 1_n \qquad (i > 0) .$$

Theorem 1. FK is a loop space for EK. In fact the twisted cartesian product $\{FK, EK, \tau\}$ has a contractible total space.

It is easy to verify that τ satisfies the conditions for a twisting function. Hence we have defined a twisted cartesian product, and therefore a principal bundle. Let T denote its total space. Note that T may be identified with FK × EK except that ∂_0 is given by

$$\partial_0(g_n, (Ek_{n-1})) = (\partial_0 g_n \cdot k_{n-1}, (b_{n-1}))$$

$$\partial_0(g_n, (s_0^i Ek_{n-i-1})) = (\partial_0 g_n, (s_0^{i-1}(Ek_{n-i-1})) \quad (i \geq 1) .$$

It will first be shown that the homology groups of T are trivial. This will be done by giving a contracting homotopy S for the chain complex C(T).

Lemma 3. Let G be the free group on generators x_α. Then the integral group ring ZG has as basis (over Z) the elements $gx_\alpha - g$, where g ranges over all elements of G; together with the element 1.

The proof is not difficult. Now define S by the rules

$$S(1_n, (b_n)) = \begin{cases} 0 & (n \text{ even}) \\ (1_{n+1}, (b_{n+1})) & (n \text{ odd}) \end{cases}$$

$$S[(g_n \cdot k_n, (b_n)) - (g_n, (b_n))]$$

$$= \sum_{i=0}^{n} (-1)^i [(s_i g_n, (s_0^i E \partial_0^i k_n)) - (s_i g_n, (b_{n+1}))]$$

$$S[(g_n, (s_0^{r-1} E k_{n-r})) - (g_n, (b_n))]$$

$$= \sum_{j=r}^{n} (-1)^j [(s_j g_n, (s_0^j E \partial_0^{j-r} k_{n-r})) - (s_j g_n, (b_{n+1}))]$$

where g_n ranges over all elements of the group FK_n.

It follows easily from Lemma 3 that the elements for which S has been defined form a basis for $C(T)$, providing that k_n, k_{n-r} are restricted to elements other than b_n, b_{n-r}. However the above rules reduce to the identity $0 = 0$ if we substitute $k_n = b_n$ or $k_{n-r} = b_{n-r}$. This shows that S is well defined.

The necessary identity $Sd + dS = 1 - \varepsilon$, where

$dx_n = \sum_{i=0}^{n} (-1)^i \partial_i x_n$ and where $\varepsilon : C(T) \to C(T)$ is the augmentation

$(\varepsilon \sum \alpha_i (g_0, b_0) = \sum \alpha_i (1_0, b_0))$ can now be verified by direct computation. Since this computation is rather long it will not be given here.

Proof that $|T|$ is simply connected. A maximal tree in the CW-complex $|T|$ will be chosen. Then $\pi_1(|T|)$ can be considered as the group with one generator corresponding to each 1-simplex not in the tree, and one relation corresponding to each 2-simplex.

As maximal tree take all 1-simplexes of the form $(s_0 g_0, (Ek_0))$. Then as generators of $\pi_1(|T|)$ we have all elements $(g_1, (Ek_0))$ such that g_1 is non-degenerate. The relation $\partial_1 x = (\partial_2 x) \cdot (\partial_0 x)$ where $x = (s_1 g_1, (s_0 E k_0))$ asserts that

$$(g_1, (Ek_0)) = (g_1, (b_1)) \cdot (s_0 \partial_0 g_1, (Ek_0))$$

$$= (g_1, (b_1)) .$$

From the 2-simplex $(s_0 g_1, (Ek_1))$ we obtain

$$(g_1, (E\partial_0 k_1)) = (s_0 \partial_1 g_1, (E\partial_1 k_1)).(g_1 k_1, (b_1))$$
$$= (g_1 k_1, (b_1)) .$$

Combining these two relations we have

$$(g_1, (b_1)) = (g_1 k_1, (b_1)) ,$$

from which it follows easily that

$$(g_1, (b_1)) = 1$$

for all g_1. In view of the first relation, this shows that $|T|$ is simply connected, and completes the proof of theorem 1.

The following theorem shows that FK is essentially unique.

Theorem 2. Any principal bundle over EK with any group complex G as fibre is induced from the above bundle by a homomorphism $FK \to G$.

Proof. We may assume that this bundle is a twisted cartesian product with twisting function $\tau:(EK)_{n+1} \to G_n$. Define the homomorphism $\bar{\tau}:FK \to G$ by $\bar{\tau}(k_n) = \tau(Ek_n)$. Since $\bar{\tau}(b_n) = \tau(Eb_n) = \tau(s_0(b_n)) = 1_n$ this defines a homomorphism. It is easy to verify that $\bar{\tau}$ commutes with the face and degeneracy operations, and induces a map between the two twisted cartesian products.

Corollary. *If G is also a loop space for EK then there is a homomorphism FK → G inducing an isomorphism between the Pontrjagin rings.*

This follows easily using [7], IV Theorem B.

Analogues of theorems 1 and 2 for the construction $F^+(K)$ can be proved using exactly the same formulas. The following shows the relationship between $F^+(K)$ and the construction of James.

Lemma 4. *If K is countable then the realization* $|F^+K|$ *is homeomorphic to the reduced product of* $|K|$.

In fact the product $(k_n, k'_n, k''_n, \ldots) \to k_n . k' . k'' . \ldots$ maps $K x \ldots x K$ into F^+K. Taking realizations we obtain a map $|K| x \ldots x |K| \to |F^+K|$. From these maps it is easy to define a map of the reduced product of $|K|$ into $|F^+K|$, and to show that it is a homeomorphism.

3. A theorem of Hilton

If A, B are semi-simplicial complexes with base points a_0, b_0 let $A \vee B$ denote the subcomplex $A \times [b_0] \cup [a_0] \times B$ of $A \times B$. Let $A \ast B$ denote the complex obtained from $A \times B$ by collapsing $A \vee B$ to a point. The notation $A^{(k)}$ will be used for the k-fold 'collapsed product' $A \ast \ldots \ast A$.

The free product $G \ast H$ of two group complexes is defined by $(G \ast H)_n = G_n \ast H_n$. There is clearly a canonical isomorphism between the group complexes $F(A \vee B)$ and $FA \ast FB$.

<u>**Lemma** 5.</u> <u>The complex</u> $F(A \vee B)$ <u>is isomorphic</u> <u>(ignoring group structure) to</u> $FA \times F(B \vee (B \ast FA))$.

In fact we will show that $F(A \vee B)$ is a split extension:

$$I \to F(B \vee (B \ast FA)) \to F(A \vee B) \to FA \to I.$$

The collapsing map $A \vee B \xrightarrow{c} A$ induces a homomorphism c' of $F(A \vee B)$ onto FA. Denote the kernel of c' by F'. The inclusion $A \xrightarrow{i} A \vee B$ induces a homomorphism $i'\colon FA \to F(A \vee B)$. Since $c'i'$ is the identity it follows that $F(A \vee B)$ is a split extension of F' by FA.

We will determine this kernel F'_n for some fixed dimension n. Let a, b, ϕ range over the n-simplexes other than the base point of A, B, and FA respectively. Then $F(A \vee B)_n$ is the free group $\{a, b\}$ and F'_n is the normal subgroup generated by the b. By the Reidemeister-Schreier theorem (see [8]) F'_n is freely generated by the elements $w(a)bw(a)^{-1}$ where $w(a)$ ranges over all elements of the free group $\{a\} = FA_n$. Thus

$$F'_n = \{b, \phi b \phi^{-1}\}.$$

Now setting $[b, \phi] = b\phi b^{-1}\phi^{-1}$ and making a simple Tietze transformation (see for example [1]) we obtain

$$F'_n = \{b, [b, \phi]\}.$$

Identify $[b, \phi]$ with the simplex $b \ast \phi$ of $B \ast F(A)$. Then we can identify F'_n with $F(B \vee (B \ast FA))$. Since this identification commutes with face and degeneracy operations, this proves Lemma

Lemma 6. *The group complex* $F(B \divideontimes FA)$ *is isomorphic to*

$$F((B \divideontimes A) \vee (B \divideontimes A \divideontimes FA)) .$$

The inclusion $A \to FA$ induces a homomorphism

$$F(B \divideontimes A) \to F(B \divideontimes FA) .$$

A homomorphism

$$F(B \divideontimes A \divideontimes FA) \to F(B \divideontimes FA)$$

is defined by

$$b \divideontimes a \divideontimes \phi \to (b \divideontimes a)(b \divideontimes \phi a)^{-1} (b \divideontimes \phi) .$$

(This is motivated by the group identity $[[b, a], \phi] = [b, a]$ $[b, \phi a]^{-1}[b, \phi]$.)

Combining these we obtain a homomorphism

$$F(B \divideontimes A) \ast \tau F(B \divideontimes A \divideontimes FA) \to F(B \divideontimes FA)$$

which is asserted to be an isomorphism.

Using the same notation as in Lemma 5 and identifying $b \divideontimes a \divideontimes \phi$ with $[[b, a], \phi]$ it is evidently sufficient to prove the following.

Lemma 7. *In the free group* $\{a, b\}$ *the subgroup freely generated by the elements* $[b, \phi]$ *is also freely generated by the elements* $[b, a]$ and $[[b, a], \phi]$.

The proof consists of a series of Tietze transformations. Details will not be given.

As a consequence of Lemma 6 we have:

Lemma 8. *For each* m *the group complex* $F(B \ast FA)$ *is isomorphic to*

$$F(B \ast A) \nmid F(B \ast A \ast A) \nmid \ldots \nmid F(B \ast A^{(m)}) \nmid$$
$$F(B \ast A^{(m)} \ast FA) .$$

Proof by induction on m. For $m = 1$ this is just Lemma 6. Given this assertion for the integer $m - 1$ it is only necessary to show that $F(B \ast A^{(m-1)} \ast FA)$ is isomorphic to $F(B \ast A^{(m)}) \nmid F(B \ast A^{(m)} \ast FA)$. But this follows immediately from Lemma 6 by substituting $B \ast A^{(m-1)}$ in place of B.

Theorem 3. *If* A *and* B *are semi-simplicial complexes with* A *connected, then there is an inclusion homomorphism*

$$F(\vee_{i=1}^{\infty} B \ast A^{(i)}) \to F(B \ast F(A))$$

which is a homotopy equivalence.

Proof. Every element of $F(\vee_{i=1}^{\infty} B \ast A^{(i)})$ is already contained in

$$F(\vee_{i=1}^{\infty} B \ast A^{(i)}) = F(B \ast A) \nmid \ldots \nmid F(B \ast A^{(m)})$$

for some m. Hence by Lemma 8 it may be identified with an element of $F(B * FA)$. Since A is connected, the 'remainder term' $B * A^{(m)} * FA$ has trivial homology groups in dimensions less than m. From this it follows easily that the above inclusion induces isomorphisms of the homotopy groups in all dimensions.

Remark. The complex B may be eliminated from Theorem 3 by taking B as the sphere S^0, and noting the identity $S^0 * K = K$.

Combining theorem 3 with Lemma 5 we obtain the following

Corollary. If A is connected then there is a homotopy equivalence

$$F(A) \times F(\vee_{i=0}^{\infty} B * A^{(i)}) \subset F(A \vee B) .$$

This corollary will be the basis for the following.

Theorem 4. Let $A_1, \ldots, A_r$ be connected complexes. Then $F(A_1 \vee \ldots \vee A_r)$ has the same homotopy type as a weak infinite cartesian product $\Pi_{i=1}^{\infty} F(A_i)$ where each A_i, $i > r$, has the form

$$A_1^{(n_1)} * \ldots * A_r^{(n_r)} .$$

The number of factors of a given form is equal to the Witt number

$$\phi(n_1, \ldots, n_r) = \frac{1}{n} \sum_{d|\delta} \frac{\mu(d)(n/d)!}{(n_1/d)! \ldots (n_r/d)!}$$

where $n = n_1 + \ldots + n_r$, $\delta = GCD(n_1, \ldots, n_r)$.

Proof. For $n = 1, 2, 3, \ldots$ define complexes A_i, to be called 'basic products of weight n' as follows, by induction on n. The given complexes $A_1, \ldots, A_r$ are the basic products of weight 1. Suppose that

$$A_1, \ldots, A_r, \ldots, A_\alpha$$

are the basic products of weight less than n. To each $i = 1, \ldots, r, \ldots, \alpha$ assume we have defined a number $e(i) < i$, where $e(1) = \ldots = e(r) = 0$. Then as basic products of weight n take all expressions $A_i \ast A_j$ where weight A_i + weight $A_j = n$ and $e(i) \leq j < i$. Call these new complexes $A_{\alpha+1}, \ldots, A_\beta$ in any order. If $A_h = A_i \ast A_j$ define $e(h) = j$. (For this discussion we must consider complexes such as $(A \ast B) \ast C$ and $A \ast (B \ast C)$ to be distinct!) This completes the construction of the A_i.

For each $m \geq 1$ define

$$R_m = F(\vee_{\substack{h \geq m \\ e(h) < m}} A_h) .$$

Thus $R_1 = F(A_1 \vee \ldots \vee A_r)$.

Lemma 9. There is a homotopy equivalence

$$F(A_m) \times R_{m+1} \subset R_m .$$

Note that $R_m = F(A_m \vee B)$, where $B = \vee_{\substack{h > m \\ e(h) < m}} A_h$.

By the corollary to theorem 3 there is a homotopy equivalence

$$(F(A_m) \times F(\vee_{i=0}^{\infty} B \ast A_m^{(i)}) \subset F(A_m \vee B) = R_m .$$

Substituting in the definition of B and using the distributive law

$$(A \vee B) \ast C = (A \ast C) \vee (B \ast C) ,$$

the second factor of the first expression becomes

$$F(\vee_{i=0}^{\infty} \vee_{\substack{h > m \\ e(h) < m}} A_h \ast A_m^{(i)}) .$$

But (filling in parentheses correctly) this is just

$$F(\vee_{\substack{h > m \\ e(h) \leq m}} A_h) = R_{m+1} ,$$

which proves Lemma 9.

Now it follows by induction that there is a homotopy equivalence

$$F(A_1) \times F(A_2) \times \dots \times F(A_m) \times R_{m+1} \subset R_1 =$$

$$F(A_1 \vee \dots \vee A_r) .$$

This defines an inclusion of the weak infinite cartesian product

$\Pi_{i=1}^{\infty} F(A_i)$ into R_1. Since $A_1, \ldots, A_r$ are connected, it follows easily that the 'remainder terms' R_m are k-connected where $k \to \infty$ as $m \to \infty$. From this it follows that the above inclusion map induces isomorphisms of the homotopy groups in all dimensions. This proves the first part of theorem 4.

Let $\phi(n_1, \ldots, n_r)$ denote the number of A_h having the form $A_1^{(n_1)} * \ldots * A_r^{(n_r)}$. To compute these numbers consider the free Lie ring L on generators $\alpha_1, \ldots, \alpha_r$. Corresponding to each 'basic product' $A_h = A_i * A_j$ define an element $\alpha_h = [\alpha_i, \alpha_j]$ of L, for $h = r+1, r+2, \ldots$. Then the elements α_h obtained in this way are exactly the standard monomials of M. Hall [2] and P. Hall [3]. M. Hall has proved that these elements form an additive basis for L.

The number of linearly independent elements of L which involve each of the generators $\alpha_1, \ldots, \alpha_r$ a given number $n_1, \ldots, n_r$ of times has been computed by Witt [9]. Since his formula is the same as that in theorem 4, this completes the proof.

In conclusion we mention one more interesting consequence of theorem 3.

<u>Theorem 5</u>. <u>If</u> A <u>is connected then the complex</u> EFA <u>has the same homotopy type as</u> $\vee_{i=1}^{\infty} EA^{(i)}$.

The proof is based on the following lemma, which depends on Theorem 1.

<u>Lemma 10</u>. <u>If</u> A <u>is connected, there is a homotopy equivalence</u>

$$EA \subset WFA .$$

In fact the inclusion is defined by $(s_0^i Ea_n) \to s_0^i(a_n, 1_{n-1}, \ldots, 1_0)$. It is easily verified that this is a map, and that it induces a map of the twisted cartesian product T into the twisted cartesian product W. Since both total spaces are acyclic, it follows from [7], IV Theorem A that the homology groups of EA map isomorphically into those of $\overline{W}FA$. Since both spaces are simply connected, this completes the proof of Lemma 10.

Now from Theorem 3 we have a homotopy equivalence

$$\overline{W}F(\vee_{i=1}^{\infty} A^{(i)}) \subset \overline{W}FFA .$$

In view of Lemma 10, and the identity

$$E(A \vee B) = EA \vee EB ,$$

this completes the proof.

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