Irreducible euclidean representations of the Fibonacci groups

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Let Γ be a *crystallographic* group of dimension n, i.e. a discrete and cocompact subgroup of the group $E(n) = O(n) \ltimes \mathbb{R}^n$ of isometries of the euclidean space $\mathbb{R}^n$. If Γ is in addition torsionfree we call it a *Bieberbach group*. In that case the orbit space $X = \mathbb{R}^n/\Gamma$ is a *flat manifold* (closed connected Riemannian manifold with sectional curvature equal to zero) and $\Gamma = \pi_1(X)$.

Let $r, n \in \mathbb{N}$. The *Fibonacci group* $F(r, n)$ is a group on n generators $a_0, \ldots, a_{n-1}$ with relations $a_i \ldots a_{i+r} = a_{i+r+1}$ where $i = 0, \ldots, n-1$ and subscripts are taken modulo n. Fibonacci groups have some interesting geometric interpretation. For example $F(2, 2n), n \geq 4$ is the fundamental group of a certain closed hyperbolic 3-manifold and $F(2, 6)$ is the fundamental group of the 3-dimensional flat manifold (see below) called Hantzsche-Wendt manifold.

In the paper [**1**] Andrzej Szczepański proves that in every odd dimension $n \geq 3$ there exist a *Hantzsche-Wendt* group, i.e. a Bieberbach group for which the holonomy group of the corresponding flat manifold is isomorphic to $\mathbb{Z}_2^{n-1}$, which is epimorphic image of the Fibonacci group $F(n-1, 2n)$.
In the review of the paper, available in MathSciNet, Juan Pablo Rossetti states that the only two Hantzsche-Wendt groups of dimension 5 are both epimorphic images of the group $F(4, 10)$. He also suggests that there may exist many epimorphisms of the type presented in the paper.

We show that for every odd n the family of subgroups of $E(n)$ which are epimorphic images of the Fibonacci group $F(n-1, 2n)$ not only includes the family of Hantzsche-Wendt groups. Groups in this family don't even have to be torsionfree and even more – they don't have to be crystallographic.

References

[1] A. Szczepański, *The Euclidean representations of the Fibonacci groups*, Q. J. Math. 52 (2001), no. 3, 385–389